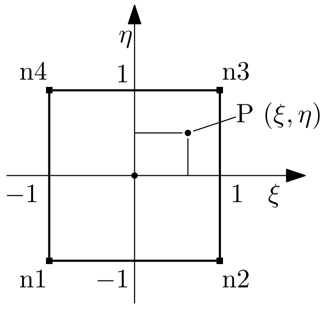
Chapter 1

Fundamentals of Finite Element Method for structural applications



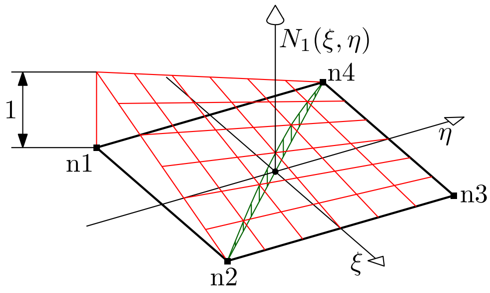


Figure 1.1: Quadrilateral elementary domain (a), and a representative weight function (b).

1.1 Preliminary results

1.1.1 Interpolation functions for the quadrilateral domain

The elementary quadrilateral domain. A quadrilateral domain is considered whose vertices are conventionally located at the $(\pm 1, \pm 1)$ points of an adimensional (ξ, η) plane coordinate system, see Figure 1.1. Scalar values f_i are associated to a set of *nodal* points $\mathrm{P}_i \equiv [\xi_i, \eta_i]$, which for the present case coincide with the quadrangle vertices, numbered as in Figure.

A $f(\xi, \eta)$ interpolation function may be devised by defining a set of nodal influence functions $N_i(\xi, \eta)$ to be employed as the coefficients (weights) of a moving weighted average

$$f(\xi, \eta) \stackrel{\text{def}}{=} \sum_i N_i(\xi, \eta) f_i \tag{1.1}$$

Requisites for such weight functions are:

- the influence of a node is unitary at its location, whereas the influence of the others locally vanishes, i.e.

  $$N_i(\xi_j, \eta_j) = \delta_{ij} \tag{1.2}$$

  where δ_{ij} is the Kronecker delta function.

- for each point of the domain, the sum of the weights is unitary

$$\sum_i N_i(\xi, \eta) = 1, \ \forall[\xi, \eta] \tag{1.3}$$

Moreover, suitable functions should be continuous and straightforwardly differentiable up to any required degree.

Low order polynomials are ideal candidates for the application; for the particular domain, the nodal weight functions may be stated as

$$N_i(\xi, \eta) \stackrel{\text{def}}{=} \frac{1}{4}\,(1 \pm \xi)\,(1 \pm \eta)\,, \tag{1.4}$$

where sign ambiguity is resolved for each i-th node by enforcing Eqn. 1.2.

The (1.3) combination of 1.4 functions turns into a general linear relation in (ξ, η) with coplanar in the ξ, η, f space – but otherwise arbitrary – nodal points.

Further generality may be introduced by *not* enforcing coplanarity.

The weight functions for the four-node quadrilateral are in fact quadratic although incomplete[1] in nature, due to the presence of the $\xi\eta$ product, and the absence of any ξ^2, η^2 term.

Each term, and the combined $f(\xi, \eta)$ function, defined as in Eqn. 1.1, behave linearly if restricted to $\xi = \text{const.}$ or $\eta = \text{const.}$ loci – namely along the four edges; quadratic behaviour may instead arise along a general direction, e.g. along the diagonals, as in Fig. 1.1b example. Such behaviour is called *bilinear*.

We now consider the $f(\xi, \eta)$ weight function partial derivatives. The partial derivative

$$\frac{\partial f}{\partial \xi} = \underbrace{\left(\frac{f_2 - f_1}{2}\right)}_{[\Delta f/\Delta \xi]_{12}} \underbrace{\left(\frac{1-\eta}{2}\right)}_{N_1+N_2} + \underbrace{\left(\frac{f_3 - f_4}{2}\right)}_{[\Delta f/\Delta \xi]_{43}} \underbrace{\left(\frac{1+\eta}{2}\right)}_{N_4+N_3} = a\eta + b \tag{1.5}$$

linearly varies from the incremental ratio value measured at the $\eta = -1$ lower edge, to the value measured at the $\eta = 1$ upper edge; the other partial derivative

$$\frac{\partial f}{\partial \eta} = \left(\frac{f_4 - f_1}{2}\right)\left(\frac{1-\xi}{2}\right) + \left(\frac{f_3 - f_2}{2}\right)\left(\frac{1+\xi}{2}\right) = c\xi + d. \tag{1.6}$$

[1] or, equivalently, *enriched linear*, as discussed above and in the following

behaves similarly, with $c = a$. However, partial derivatives in ξ, η remain constant along the corresponding differentiation direction [2].

The general quadrilateral domain. The interpolation functions introduced above for the natural quadrilateral may be profitably employed in defining a coordinate mapping between a general quadrangular domain – see Fig. 1.2a – and its reference counterpart, see Figures 1.1 and 1.2b.

In particular, we first define the $\underline{\xi}_i \mapsto \underline{x}_i$ coordinate mapping for the four vertices[3] alone, where ξ, η are the reference (or natural, or elementary) coordinates and x, y are their physical counterpart.

Then, a mapping for the inner points may be derived by interpolation, namely

$$\underline{x} = \underline{m}\left(\underline{\xi}\right) = \sum_{i=1}^{4} N_i\left(\underline{\xi}\right)\underline{x}_i \tag{1.7}$$

The availability of an inverse $\underline{m}^{-1} : \underline{x} \mapsto \underline{\xi}$ mapping is not granted; in particular, a closed form representation for such inverse is not generally available[4].

In the absence of an handy inverse mapping function, it is convenient to reinstate the interpolation procedure obtained for the natural domain, i.e.

$$f(\xi, \eta) \overset{\text{def}}{=} \sum_i N_i(\xi, \eta) f_i \tag{1.8}$$

The four f_i nodal values are interpolated based on the *natural* ξ, η coordinates of an inner P point, and not as a function of its physical x, y coordinates, that are never promoted to the independent variable role.

As already mentioned, the $\underline{m}$ mapping behaves linearly along η =const. and ξ =const. one dimensional subdomains, and in particular along

[2] The relevance of such partial derivative orders will appear clearer to the reader once the strain field will have been derived in paragraph XXX.

[3] The condensed notation $\underline{\xi}_i \equiv (\xi_i, \eta_i)$, $\underline{x}_i \equiv (x_i, y_i)$ for coordinate vectors is employed.

[4] Inverse relations are derived in [1], which however are case-defined and based on a selection table; for a given $\bar{\underline{x}}$ physical point, however, Newton-Raphson iterations rapidly converge to the $\bar{\underline{\xi}} = \underline{m}^{-1}\left(\bar{\underline{x}}\right)$ solution if the centroid is chosen for algorithm initialization, see Section XXX

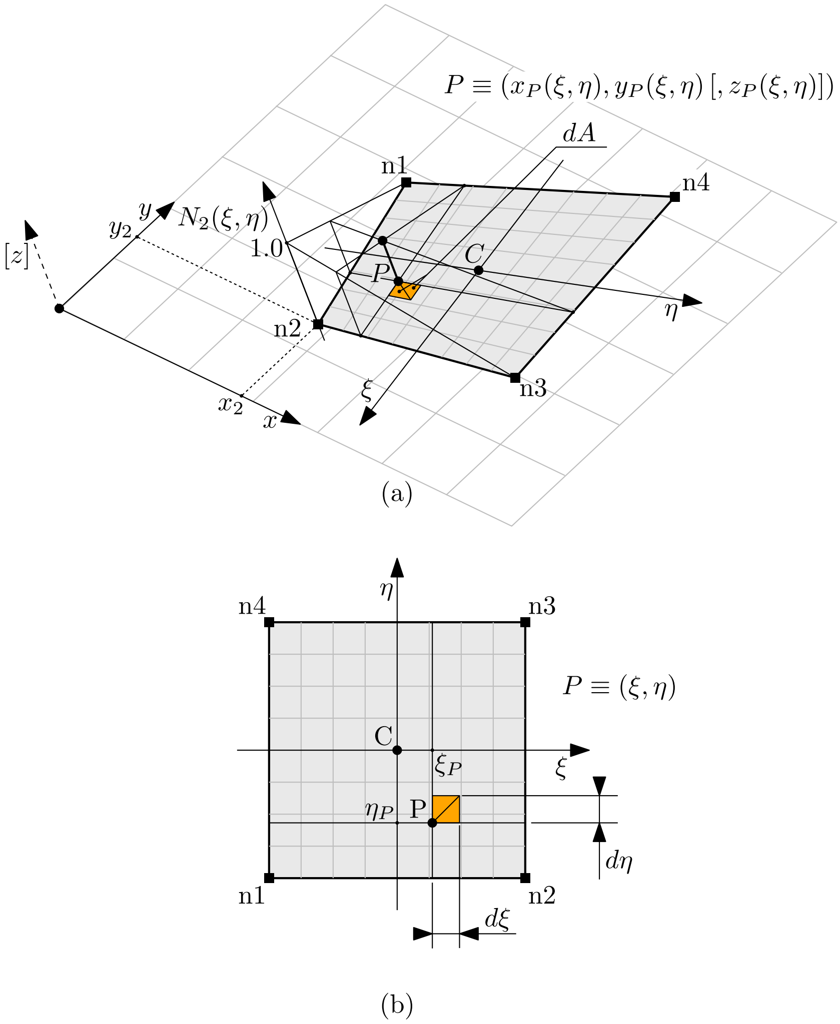


Figure 1.2: Quadrilateral general domain, (a), and its reference counterpart (b). If the general quadrangle is defined within a spatial environment, and not as a figure lying on the xy plane, limited z_i offsets are allowed at nodes with respect to such plane, which are not considered in Figure.

the quadrangle edges[5]; the inverse mapping $\underline{\mathrm{m}}^{-1}$ exists along these line segments under the further condition that their length is nonzero[6], and it is a linear function[7] . Being a composition of linear functions, the interpolation function $f(\underline{\mathrm{m}}^{-1}(x,y))$ is also linear along the aforementioned subdomains, and in particular along the quadrangle edges.

The directional derivatives of f with respect to x or y are obtained based the indirect relation

$$\begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{\underline{\underline{\mathrm{J}}}'(\xi,\eta)} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \tag{1.9}$$

.

The function derivatives with respect to ξ, η are obtained as

$$\begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = \sum_i \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} f_i. \tag{1.10}$$

The *transposed* Jacobian matrix of the mapping function that appears in 1.9 is

$$\underline{\underline{\mathrm{J}}}'(\xi,\eta) = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{1.11}$$

$$= \sum_i \left(\begin{bmatrix} \frac{\partial N_i}{\partial \xi} & 0 \\ \frac{\partial N_i}{\partial \eta} & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 & \frac{\partial N_i}{\partial \xi} \\ 0 & \frac{\partial N_i}{\partial \eta} \end{bmatrix} y_i \right) \tag{1.12}$$

If the latter matrix is assumed nonsingular – condition, this, that pairs the bijective nature of the $\underline{\mathrm{m}}$ mapping, equation 1.9 may be

[5]see paragraph XXX

[6]The case exists of an edge whose endpoints are superposed, i.e. the edge collapses to a point.

[7]A constructive proof may be defined for each edge by retrieving the non-uniform amongst the ξ, η coordinates, namely λ, as the ratio

$$\lambda = 2\frac{(x_Q - x_i)(x_j - x_i) + (y_Q - y_i)(y_j - y_i)}{(x_j - x_i)^2 + (y_j - y_i)^2} - 1,$$

where Q is a generic point along the edge, and i,j are the two subdomain endpoints at which λ equates -1 and $+1$, respectively. A similar function may be defined for any constant ξ, η segment.

inverted, thus leading to the form

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \left(\underline{\underline{J}}' \right)^{-1} \begin{bmatrix} \dots & \frac{\partial N_i}{\partial \xi} & \dots \\ \dots & \frac{\partial N_i}{\partial \eta} & \dots \end{bmatrix} \begin{bmatrix} \vdots \\ f_i \\ \vdots \end{bmatrix}, \tag{1.13}$$

where the inner mechanics of the matrix-vector product are appointed for the Eq. 1.10 summation.

1.1.2 Gaussian quadrature rules for some relevant domains.

Reference one dimensional domain. The gaussian quadrature rule for approximating the definite integral of a $f(\xi)$ function over the $[-1, 1]$ reference interval is constructed as the customary weighted sum of internal function samples, namely

$$\int_{-1}^{1} f(\xi) d\xi \approx \sum_{i=1}^{n} f(\xi_i) w_i; \tag{1.14}$$

Its peculiarity is to employ location-weight pairs (ξ_i, w_i) that are optimal with respect to the polynomial class of functions. Nevertheless, such choice has revealed itself to be robust enough for for a more general employment.

Let's consider a m-th order polynomial

$$p(\xi) \stackrel{\text{def}}{=} a_m \xi^m + a_{m-1} \xi^{m-1} + ... + a_1 \xi + a_0$$

whose exact integral is

$$\int_{-1}^{1} p(\xi) d\xi = \sum_{j=0}^{m} \frac{(-1)^j + 1}{j + 1} a_j$$

The integration residual between the exact definite integral and the weighted sample sum is defined as

$$r\left(a_j, (\xi_i, w_i)\right) \stackrel{\text{def}}{=} \sum_{i=1}^{n} p(\xi_i) w_i - \int_{-1}^{1} p(\xi) d\xi \tag{1.15}$$

The optimality condition is stated as follows: the quadrature rule involving n sample points (ξ_i, w_i), $i = 1 \dots n$ is optimal for the m-th order polynomial if a) the integration residual is null for general

a_j values , and b) such condition does not hold for any lower-order sampling rule.

Once observed that the zero residual requirement is satisfied by any sampling rule if the polynomial a_j coefficients are all null, condition a) may be enforced by imposing that such zero residual value remains constant with varying a_j terms, i.e.

$$\left\{ \frac{\partial r\left(a_j, (\xi_i, w_i)\right)}{\partial a_j} = 0, \quad j = 0 \dots m \right. \tag{1.16}$$

A system of $m+1$ polynomial equations of degree $m-1$ is hence obtained in the $2n$ (ξ_i, w_i) unknowns; in the assumed absence of redundant equations, solutions do not exist if the constraints outnumber the unknowns, i.e. $m > 2n-1$. Limiting our discussion to the threshold condition $m = 2n-1$, an attentive algebraic manipulation of Eqns. 1.16 may be performed in order to extract the (ξ_i, w_i) solutions, which are unique apart from the pair permutations[8].

Eqns. 1.16 solutions are reported in Table 1.1 for quadrature rules with up to $n = 4$ sample points[9].

[8] In this note, location-weight pairs are obtained for the gaussian quadrature rule of order $n = 2$, aiming at illustrating the general procedure. The general $m = 2n-1 = 3$rd order polynomial is stated in the form

$$p(\xi) = a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0, \quad \int_{-1}^{1} p(\xi)d\xi = \frac{2}{3}a_2 + 2a_0,$$

whereas the integral residual is

$$r = a_3\left(w_1\xi_1^3 + w_2\xi_2^3\right) + a_2\left(w_1\xi_1^2 + w_2\xi_2^2 - \frac{2}{3}\right) + a_1\left(w_1\xi_1 + w_2\xi_2\right) + a_0\left(w_1 + w_2 - 2\right)$$

Eqns 1.16 may be derived as

$$\left\{ \begin{array}{lr} 0 = \frac{\partial r}{\partial a_3} = w_1\xi_1^3 + w_2\xi_2^3 & (e_1) \\ 0 = \frac{\partial r}{\partial a_2} = w_1\xi_1^2 + w_2\xi_2^2 - \frac{2}{3} & (e_2) \\ 0 = \frac{\partial r}{\partial a_1} = w_1\xi_1 + w_2\xi_2 & (e_3) \\ 0 = \frac{\partial r}{\partial a_0} = w_1 + w_2 - 2 & (e_4) \end{array} \right.$$

which are independent of the a_j coefficients.

By composing $\left(e_1 - \xi_1^2 e_3\right)/(w_2\xi_2)$ it is obtained that $\xi_2^2 = \xi_1^2$; e_2 may then be written as $(w_1 + w_2)\xi_1^2 = 2/3$, and then as $2\xi_1^2 = 2/3$, according to e_4. It derives that $\xi_{1,2} = \pm 1/\sqrt{3}$. Due to the opposite nature of the roots, e_3 implies $w_2 = w_1$, relation, this, that turns e_4 into $2w_1 = 2w_2 = 2$, and hence $w_{1,2} = 1$.

[9]see `https://pomax.github.io/bezierinfo/legendre-gauss.html` for higher

n	ξ_i	w_i
1	0	2
2	$\pm\frac{1}{\sqrt{3}}$	1
3	0	$\frac{8}{9}$
	$\pm\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
4	$\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18+\sqrt{30}}{36}$
	$\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18-\sqrt{30}}{36}$

Table 1.1: Integration points for the lower order gaussian quadrature rules.

It is noted that the integration points are symmetrically distributed with respect to the origin, and that the function is never sampled at the $\{-1, 1\}$ extremal points.

General one dimensional domain. The extension of the one dimensional quadrature rule from the reference domain $[-1, 1]$ to a general $[a, b]$ domain is pretty straightforward, requiring just a change of integration variable to obtain the following

$$\begin{aligned}\int_a^b f(x)dx &= \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b+a}{2}+\frac{b-a}{2}\xi\right)d\xi,\\ &\approx \frac{b-a}{2}\sum_{i=1}^{n} f\left(\frac{b+a}{2}+\frac{b-a}{2}\xi_i\right)w_i.\end{aligned}$$

Reference quadrangular domain. A quadrature rule for the reference quadrangular domain of Figure 1.1a may be derived by nesting the quadrature rule defined for the reference interval, see Eqn. 1.14, thus obtaining

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta \approx \sum_{i=1}^{p}\sum_{j=1}^{q} f\left(\xi_i,\eta_j\right)w_i w_j \tag{1.17}$$

order gaussian quadrature rule sample points.

where (ξ_i, w_i) and (η_j, w_j) are the coordinate-weight pairs of the two quadrature rules of p and q order, respectively, employed for spanning the two coordinate axes. The equivalent notation

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta \approx \sum_{l=1}^{pq} f\left(\underline{\xi}_l\right) w_l \tag{1.18}$$

emphasises the characteristic nature of the pq point/weight pairs for the domain and for the quadrature rule employed; a general integer bijection[10] $\{1 \dots pq\} \leftrightarrow \{1 \dots p\} \times \{1 \dots q\}$, $l \leftrightarrow (i, j)$ may be utilized to formally derive the two-dimensional quadrature rule pairs

$$\underline{\xi}_l = \left(\xi_i, \eta_j\right), \quad w_l = w_i w_j, \quad l = 1 \dots pq \tag{1.19}$$

from their uniaxial counterparts.

General quadrangular domain. The rectangular infinitesimal area $dA_{\xi\eta} = d\xi d\eta$ in the neighborhood of a ξ_P, η_P location, see Figure 1.2b, is mapped to the quadrangle of Figure 1.2a, which is composed by the two triangular areas

$$\begin{aligned} dA_{xy} = \quad & \frac{1}{2!}\begin{vmatrix} 1 & x(\xi_P, \eta_P) & y(\xi_P, \eta_P) \\ 1 & x(\xi_P + d\xi, \eta_P) & y(\xi_P + d\xi, \eta_P) \\ 1 & x(\xi_P, \eta_P + d\eta) & y(\xi_P, \eta_P + d\eta) \end{vmatrix} + \\ & + \frac{1}{2!}\begin{vmatrix} 1 & x(\xi_P + d\xi, \eta_P + d\eta) & y(\xi_P + d\xi, \eta_P + d\eta) \\ 1 & x(\xi_P, \eta_P + d\eta) & y(\xi_P, \eta_P + d\eta) \\ 1 & x(\xi_P + d\xi, \eta_P) & y(\xi_P + d\xi, \eta_P) \end{vmatrix}. \end{aligned} \tag{1.20}$$

[10] e.g.

$$\{i-1; j-1\} = (l-1) \bmod (p, q), \quad l - 1 = (j-1)q + (i-1)$$

where the operator

$$\{a_n; \dots; a_3; a_2; a_1\} = m \bmod (b_n, \dots, b_3, b_2, b_1)$$

consists in the extraction of the n least significant a_i digits of a mixed radix representation of the integer m with respect to the sequence of b_i bases, with $i = 1 \dots n$.

The determinant formula for the area of a triangle, shown below along with its n-dimensional symplex hypervolume generalization,

$$\mathcal{A} = \frac{1}{2!}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, \quad \mathcal{H} = \frac{1}{n!}\begin{vmatrix} 1 & \underline{x}_1 \\ 1 & \underline{x}_2 \\ \vdots & \vdots \\ 1 & \underline{x}_{n+1} \end{vmatrix} \tag{1.21}$$

has been employed above.

By operating a local multivariate linearization of the 1.20 matrix terms, the relation

$$\begin{aligned} dA_{xy} \approx \; & \frac{1}{2!}\begin{vmatrix} 1 & x & y \\ 1 & x + x_{,\xi}d\xi & y + y_{,\xi}d\xi \\ 1 & x + x_{,\eta}d\eta & y + y_{,\eta}d\eta \end{vmatrix} + \\ & + \frac{1}{2!}\begin{vmatrix} 1 & x + x_{,\xi}d\xi + x_{,\eta}d\eta & y + y_{,\xi}d\xi + y_{,\eta}d\eta \\ 1 & x + x_{,\eta}d\eta & y + y_{,\eta}d\eta \\ 1 & x + x_{,\xi}d\xi & y + y_{,\xi}d\xi \end{vmatrix} \end{aligned}$$

is obtained, where $x, y, x_{,\xi}, x_{,\eta}, y_{,\xi}$, and $y_{,\eta}$ are the x, y functions and their first order partial derivatives, sampled at the (ξ_P, η_P) point; infinitesimal terms of order higher than $d\xi, d\eta$ are neglected.

After some matrix manipulations[11], the following expression is obtained

$$dA_{xy} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & x_{,\xi} & y_{,\xi} \\ 0 & x_{,\eta} & y_{,\eta} \end{vmatrix} d\xi d\eta = \underbrace{\begin{vmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{vmatrix}}_{|J^{\mathrm{T}}(\xi_P, \eta_P)|} dA_{\xi\eta} \tag{1.22}$$

that equates the ratio of the mapped and of the reference areas to the determinant of the transformation (transpose) Jacobian matrix[12].

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[11] For both the determinants, the first column is multiplied by x_P and subtracted to the second column, and then subtracted to the third column once multiplied by y_P. The first row is then subtracted to the others. On the second determinant alone, both the second and the third columns are changed in sign; then, the second and the third rows are summed to the first. The two determinants are now formally equal, and the two $1/2$ multipliers are summed to provide unity. The $d\xi$ and the $d\eta$ factors may then be collected from the second and the third rows, respectively.

[12] The Jacobian matrix for a general $\underline{\xi} \mapsto \underline{x}$ mapping is in fact defined according to

$$[J(\underline{\xi}_P)]_{ij} \overset{\text{def}}{=} \left.\frac{\partial x_i}{\partial \xi_j}\right|_{\underline{\xi} = \underline{\xi}_P} \qquad i, j = 1 \dots n$$

being i the generic matrix term row index, and j the column index

After the preparatory passages above, we obtain

$$\iint_{A_{xy}} g(x,y)dA_{xy} = \iint_{-1}^{1} g\left(x\left(\xi,\eta\right), y\left(\xi,\eta\right)\right)\left|J(\xi,\eta)\right| d\xi d\eta, \tag{1.23}$$

thus reducing the quadrature over a general domain to its reference domain counterpart, which has been discussed in the paragraph above.

Based on Eqn. 1.18, the quadrature rule

$$\iint_{A_{xy}} g(\underline{\mathrm{x}})dA_{xy} \approx \sum_{l=1}^{pq} g\left(\underline{\mathrm{x}}\left(\underline{\xi}_{l}\right)\right)\left|J(\underline{\xi}_{l})\right| w_l \tag{1.24}$$

is derived, stating that the definite integral of a g integrand over a quadrangular domain pertaining to the physical x, y plane (x, y are dimensional quantities, namely lengths) may be approximated as follows:

1. a reference-to-physical domain mapping is defined, that is based on the vertex physical coordinate interpolation;
2. the function is sampled at the physical locations that are the images of the Gaussian integration points previously obtained for the reference domain;
3. a weighted sum of the collected samples is performed, where the weights consist in the product of i) the adimensional w_l Gauss point weight (suitable for integrating on the reference domain), and ii) a dimensional area scaling term, that equals the determinant of the transformation Jacobian matrix, locally evaluated at the Gauss points.

1.2 Basic theory of plates

P displacement components as a function of the Q reference point motion.

$$u_P = u + z\,(1 + \tilde{\epsilon}_z)\, sin\,\phi \tag{1.25}$$
$$v_P = v - z\,(1 + \tilde{\epsilon}_z)\, sin\,\theta \tag{1.26}$$
$$w_P = w + z\,((1 + \tilde{\epsilon}_z)\, cos(\phi)\, cos(\theta) - 1) \tag{1.27}$$

$$\tilde{\epsilon}(z) = \frac{1}{z}\int_0^z \epsilon_z d\varsigma \tag{1.28}$$
$$= \frac{1}{z}\int_0^z \left(-\nu\epsilon_x - \nu\epsilon_y\right) d\varsigma \tag{1.29}$$

P displacement components as a function of the Q reference point motion, linarized with respect to the small rotations and small strain hypotheses.

$$u_P = u + z\phi \tag{1.30}$$
$$v_P = v - z\theta \tag{1.31}$$
$$w_P = w \tag{1.32}$$

Relation between the normal displacement x, y gradient (i.e. the deformed plate slope), the rotations and the out-of-plane, interlaminar, averaged shear strain components.

$$\frac{\partial w}{\partial x} = \bar{\gamma}_{zx} - \phi \tag{1.33}$$
$$\frac{\partial w}{\partial y} = \bar{\gamma}_{yz} + \theta \tag{1.34}$$

Strains at point P.

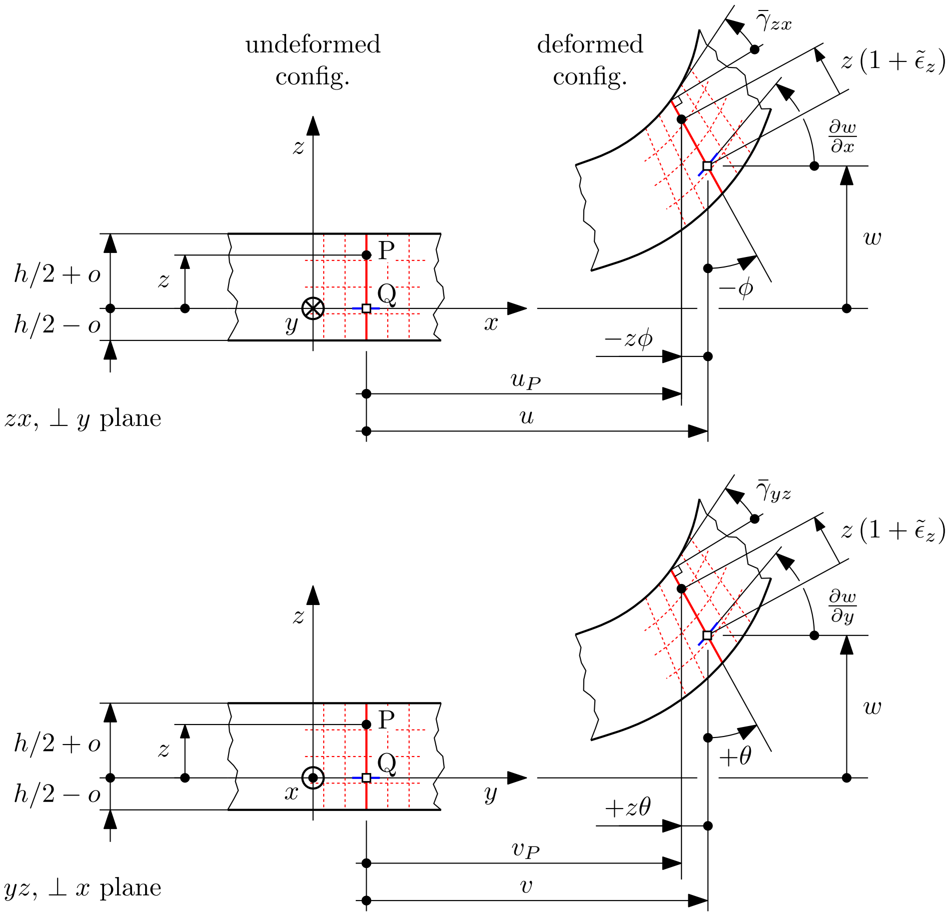


Figure 1.3: Relevant dimensions for describing the deformable plate kinematics.

$$\epsilon_x = \frac{\partial u_P}{\partial x} = \frac{\partial u}{\partial x} + z\frac{\partial \phi}{\partial x} \tag{1.35}$$

$$\epsilon_y = \frac{\partial v_P}{\partial y} = \frac{\partial v}{\partial y} - z\frac{\partial \theta}{\partial y} \tag{1.36}$$

$$\gamma_{xy} = \frac{\partial u_P}{\partial y} + \frac{\partial v_P}{\partial x} \tag{1.37}$$

$$= \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + z\left(+\frac{\partial \phi}{\partial y} - \frac{\partial \theta}{\partial x}\right) \tag{1.38}$$

Generalized plate strains: membrane strains.

$$\underline{\bar{\epsilon}} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} \bar{\epsilon}_x \\ \bar{\epsilon}_y \\ \bar{\gamma}_{xy} \end{pmatrix} \tag{1.39}$$

Generalized plate strains: curvatures.

$$\underline{\kappa} = \begin{pmatrix} +\frac{\partial \phi}{\partial x} \\ -\frac{\partial \theta}{\partial y} \\ +\frac{\partial \phi}{\partial y} - \frac{\partial \theta}{\partial x} \end{pmatrix} = \begin{pmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix} \tag{1.40}$$

Compact form for the strain components at P.

$$\underline{\epsilon} = \underline{\bar{\epsilon}} + z\,\underline{\kappa} \tag{1.41}$$

Hook law for an isotropic material, under plane stress conditions.

$$\underline{\underline{D}} = \frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \tag{1.42}$$

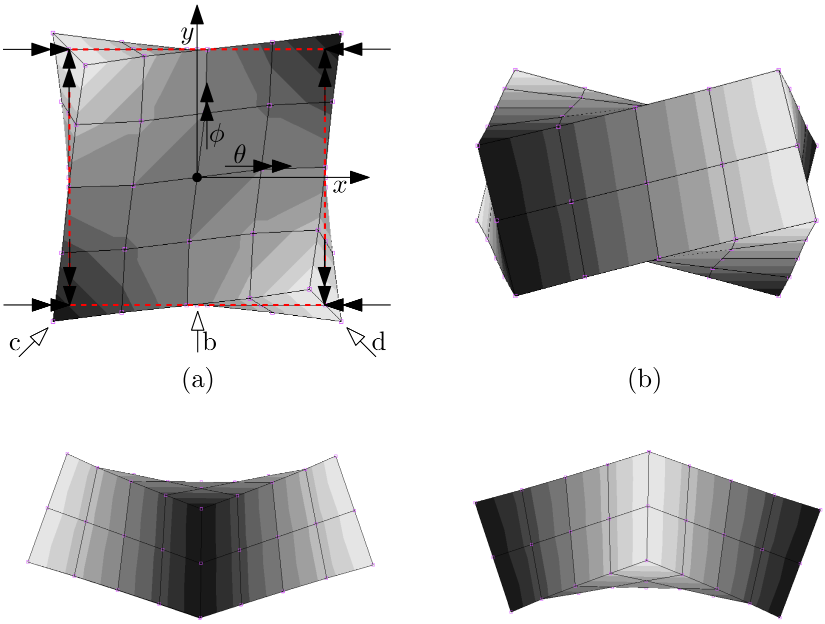
Normal components for stress and strain, the latter for the isotropic material case only.

$$\sigma_z = 0 \tag{1.43}$$

$$\epsilon_z = -\nu\left(\epsilon_x + \epsilon_y\right) \tag{1.44}$$

Stresses at P.

$$\underline{\sigma} = \underline{\underline{D}}\,\underline{\epsilon} = \underline{\underline{D}}\,\underline{\bar{\epsilon}} + z\,\underline{\underline{D}}\,\underline{\kappa} \tag{1.45}$$



(a) (b)

(c) (d)

Figure 1.4: Positive κ_{xy} torsional curvature for the plate element. Subfigure (a) shows the positive γ_{xy} shear strain at the upper surface, the (in-plane) undeformed midsurface, and the negative γ_{xy} at the lower surface; the point of sight related to subfigures (b) to (d) are also evidenced. θ and ϕ rotation components decrease with x and increase with y, respectively, thus leading to positive κ_{xy} contributions. As shown in subfigures (c) and (d), the torsional curvature of subfigure (b) evolves into two anticlastic bending curvatures if the reference system is aligned with the square plate element diagonals, and hence rotated by 45° with respect to z.

Membrane (direct and shear) stress resultants (shear flows).

$$\underline{q} = \begin{pmatrix} q_x \\ q_y \\ q_{xy} \end{pmatrix} = \int_h \underline{\sigma}\, dz \tag{1.46}$$

$$= \underbrace{\int_h \underline{\underline{D}}\, dz}_{\underline{\underline{A}}}\ \bar{\underline{\epsilon}} + \underbrace{\int_h \underline{\underline{D}}\, z dz}_{\underline{\underline{B}}}\ \underline{\kappa} \tag{1.47}$$

Bending and torsional moment stress resultants (moment flows).

$$\underline{m} = \begin{pmatrix} m_x \\ m_y \\ m_{xy} \end{pmatrix} = \int_h \underline{\sigma}\, dz \tag{1.48}$$

$$= \underbrace{\int_h \underline{\underline{D}}\, z dz}_{\underline{\underline{B}} \equiv \underline{\underline{B}}^{T}}\ \bar{\underline{\epsilon}} + \underbrace{\int_h \underline{\underline{D}}\, z^2 dz}_{\underline{\underline{C}}}\ \underline{\kappa} \tag{1.49}$$

Cumulative generalized strain - stress relations for the plate (or for the laminate)

$$\begin{pmatrix} \underline{q} \\ \underline{m} \end{pmatrix} = \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{B}}^{T} & \underline{\underline{C}} \end{pmatrix} \begin{pmatrix} \bar{\underline{\epsilon}} \\ \underline{\kappa} \end{pmatrix} \tag{1.50}$$

Hook law for the orthotropic material in plane stress conditions, with respect to principal axes of orthotropy;

$$\underline{\underline{D}}_{123} = \begin{pmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{pmatrix} \tag{1.51}$$

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{pmatrix} = \underline{\underline{T}}_1 \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \qquad \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{pmatrix} = \underline{\underline{T}}_2 \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} \tag{1.52}$$

where

$$\underline{\underline{T}}_1 = \begin{pmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{pmatrix} \tag{1.53}$$

$$\underline{\underline{T}}_2 = \begin{pmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{pmatrix} \tag{1.54}$$

α is the angle between 1 and x;

$$m = cos(\alpha) \qquad\qquad n = sin(\alpha) \tag{1.55}$$

The inverse transformations may be obtained based on the relations

$$\underline{\underline{T}}_1^{-1}(+\alpha) = \underline{\underline{T}}_1(-\alpha) \qquad\qquad \underline{\underline{T}}_2^{-1}(+\alpha) = \underline{\underline{T}}_2(-\alpha) \tag{1.56}$$

Finally

$$\underline{\sigma} = \underline{\underline{D}}\,\underline{\epsilon} \qquad\qquad \underline{\underline{D}} \equiv \underline{\underline{D}}_{xyz} = \underline{\underline{T}}_1^{-1}\underline{\underline{D}}_{123}\underline{\underline{T}}_2 \tag{1.57}$$

Notes:

- Midplane is ill-defined if the material distribution is not symmetric; the geometric midplane (i.e. the one obtained by ignoring the material distribution) exhibits no relevant properties in general. Its definition is nevertheless pretty straighforward.
- If the unsimmetric laminate is composed by isotropic layers, a reference plane may be obtained for which the $\underline{\underline{B}}$ membrane-to-bending coupling matrix vanishes; a similar condition may not be verified in the presence of orthotropic layers.
- Thermally induced distortion is not self-compensated in an unsymmetric laminate even if the temperature is held constant through the thickness.

1.3 The bilinear isoparametric shear-deformable shell element

> This is a four-node, thick-shell element with global displacements and rotations as degrees of freedom. Bilinear interpolation is used for the coordinates, displacements and the rotations. The membrane strains are obtained from the displacement field; the curvatures from the rotation field. The transverse shear strains are calculated at the middle of the edges and interpolated to the integration points. In this way, a very efficient and simple element is obtained which exhibits correct behavior in the limiting case of thin shells. The element can be used in curved shell analysis as well as in the analysis of complicated plate structures. For the latter case, the element is easy to use since connections between intersecting plates can be modeled without tying. Due to its simple formulation when compared to the standard higher order shell elements, it is less expensive and, therefore, very attractive in nonlinear analysis. The element is not very sensitive to distortion, particularly if the corner nodes lie in the same plane. All constitutive relations can be used with this element.
>
> — MSC.Marc 2013.1 Documentation, vol. B, Element library.

Once recalled the required algebraic paraphernalia, the definition of a bilinear quadrilateral shear-deformable isoparametric shell element is straightforward.

The quadrilateral element geometry is defined by the position in space of its four vertices, which constitute the set of *nodal points*, or *nodes*, i.e. the set of locations at which field components are primarily, parametrically, defined; interpolation is employed in deriving the field values elsewhere.

A suitable interpolation scheme, named *bilinear*, has been introduced in paragraph 1.1.1; the related functions depend on the normalized coordinate pair $\xi, \eta \in [-1, 1]$ that spans the elementary quadrilateral of Figure 1.1.

A global reference system $\mathrm{O}XYZ$ is employed for concurrently dealing with multiple elements (i.e. at a whole model scale); a more convenient, local Cxyz reference system, z being locally normal to the shell, is used when a single element is under scrutiny – e.g. in the current paragraph.

Nodal coordinates define the element initial, undeformed, geometry[13] of the portion of shell reference surface pertaining to the currente element; spatial coordinates for each other element point may be retrieved by interpolation based on the associated pair of natural ξ, η coordinates.

In particular, the C centroid is the image within the physical space of the $\xi = 0, \eta = 0$ natural coordinate system origin.

The in-plane orientation of the local Cxyz reference system is somewhat arbitrary and implementation-specific; the MSC.Marc approach is used as an example, and it is described in the following. The in-plane x, y axes are tentatively defined[14] based on the physical directions that are associated with the ξ, η natural axes, i.e. the oriented segments spanning a) from the midpoint of the n4-n1 edge to the midpoint of the n2-n3 edge, and b) from the midpoint of the n1-n2 edge to the midpoint of the n3-n4 edge, respectively; however, these two tentative axes are not mutually orthogonal in general. The mutual Cxy angle is then amended by rotating those interim axes with respect to a third, binormal axis Cz, while preserving their initial bisectrix.

The resulting quadrilateral shell element is in fact initially flat, apart from a (suggestedly limited) anticlastic curvature of the element diagonals, that is associated to the quadratic $\xi\eta$ term of the interpolation functions. The curve nature of a thin wall midsurface is thus represented by recurring to a plurality of basically flat, but mutually angled elements.

The element degrees of freedom consist in the displacements and the rotations of the four quadrilateral vertices, i.e. *nodes*.

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[13]They are however continuosly updated within most common nonlinear analysis frameworks, where *initial* usually refers to the last computed, aka *previous* step of an iterative scheme.

[14]The MSC.Marc element library documentation defines them as a normalized form of the

$$\left(\frac{\partial X}{\partial \xi}, \frac{\partial Y}{\partial \xi}, \frac{\partial Z}{\partial \xi},\right)\Big|_{\xi=0,\eta=0}, \left(\frac{\partial X}{\partial \eta}, \frac{\partial Y}{\partial \eta}, \frac{\partial Z}{\partial \eta},\right)\Big|_{\xi=0,\eta=0},$$

vectors, which are evaluated at the centroid. The two definitions may be proved equivalent based on the bilinear interpolation properties.

By interpolating the nodal values, displacement and rotation functions may be derived along the element as

$$\begin{bmatrix} u(\xi,\eta) \\ v(\xi,\eta) \\ w(\xi,\eta) \end{bmatrix} = \sum_{i=1}^{4} N_i(\xi,\eta) \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} \tag{1.58}$$

$$\begin{bmatrix} \theta(\xi,\eta) \\ \varphi(\xi,\eta) \\ \psi(\xi,\eta) \end{bmatrix} = \sum_{i=1}^{4} N_i(\xi,\eta) \begin{bmatrix} \theta_i \\ \varphi_i \\ \psi_i \end{bmatrix} \tag{1.59}$$

with $i = 1 \dots 4$ cycling along the element nodes.

By recalling Eqn. 1.13, we have e.g.

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \underbrace{\left(\underline{\underline{\mathrm{J}}}'\right)^{-1} \begin{bmatrix} \dots & \frac{\partial N_i}{\partial \xi} & \dots \\ \dots & \frac{\partial N_i}{\partial \eta} & \dots \end{bmatrix}}_{\underline{\underline{\nabla}}(\xi,\eta)} \begin{bmatrix} \vdots \\ u_i \\ \vdots \end{bmatrix} \tag{1.60}$$

for the x-oriented displacement component; the isoparametric differential operator $\underline{\underline{\nabla}}(\xi,\eta)$ is also defined that extract the x, y directional derivatives from the nodal values of a given field component.

We now collect within the five column vectors

$$\underline{\mathrm{u}} = \begin{bmatrix} \vdots \\ u_i \\ \vdots \end{bmatrix}, \ \underline{\mathrm{v}} = \begin{bmatrix} \vdots \\ v_i \\ \vdots \end{bmatrix}, \ \underline{\mathrm{w}} = \begin{bmatrix} \vdots \\ w_i \\ \vdots \end{bmatrix}, \ \underline{\theta} = \begin{bmatrix} \vdots \\ \theta_i \\ \vdots \end{bmatrix}, \ \underline{\varphi} = \begin{bmatrix} \vdots \\ \varphi_i \\ \vdots \end{bmatrix} \tag{1.61}$$

the nodal degrees of freedom; the $\underline{\psi}$ vector associated with the drilling degree of freedom is omitted.

A block defined $Q(\xi,\eta)$ matrix is thus obtained that cumulatively relates the in-plane displacement component derivatives to the associated nodal values

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{\underline{\nabla}}(\xi,\eta) & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{\nabla}}(\xi,\eta) \end{bmatrix}}_{\underline{\underline{\mathrm{Q}}}(\xi,\eta)} \begin{bmatrix} \underline{\mathrm{u}} \\ \underline{\mathrm{v}} \end{bmatrix} \tag{1.62}$$

An equivalent relation may then be obtained for the rotation field

$$\begin{bmatrix} \frac{\partial\theta}{\partial x} \\ \frac{\partial\theta}{\partial y} \\ \frac{\partial\varphi}{\partial x} \\ \frac{\partial\varphi}{\partial y} \end{bmatrix} = \underline{\underline{\mathrm{Q}}}(\xi,\eta)\begin{bmatrix} \underline{\theta} \\ \underline{\phi} \end{bmatrix} \tag{1.63}$$

By making use of two auxiliary matrices H' and H'' that collect the $0, \pm 1$ coefficients in Eqns. 1.39 and 1.40, we obtain

$$\begin{bmatrix} \bar{\epsilon}_x \\ \bar{\epsilon}_y \\ \bar{\gamma}_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & +1 & +1 & 0 \end{bmatrix}}_{\underline{\underline{\mathrm{H}}}'} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \underline{\underline{\mathrm{H}}}'\,\underline{\underline{\mathrm{Q}}}(\xi,\eta)\begin{bmatrix} \underline{\mathrm{u}} \\ \underline{\mathrm{v}} \end{bmatrix} \tag{1.64}$$

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & +1 \end{bmatrix}}_{\underline{\underline{\mathrm{H}}}''} \begin{bmatrix} \frac{\partial\theta}{\partial x} \\ \frac{\partial\theta}{\partial y} \\ \frac{\partial\varphi}{\partial x} \\ \frac{\partial\varphi}{\partial y} \end{bmatrix} = \underline{\underline{\mathrm{H}}}''\,\underline{\underline{\mathrm{Q}}}(\xi,\eta)\begin{bmatrix} \underline{\theta} \\ \underline{\varphi} \end{bmatrix} \tag{1.65}$$

The in plane strain tensor at each ξ, η, z point along the element may then be derived according to Eqn. 1.41 as a (linear) function of the nodal degrees of freedom

$$\underline{\epsilon}(\xi,\eta,z) = \begin{bmatrix} \underline{\underline{\mathrm{H}}}'\,\underline{\underline{\mathrm{Q}}}(\xi,\eta) & \underline{\underline{0}} & z\,\underline{\underline{\mathrm{H}}}''\,\underline{\underline{\mathrm{Q}}}(\xi,\eta) \end{bmatrix} \begin{bmatrix} \underline{\mathrm{u}} \\ \underline{\mathrm{v}} \\ \underline{\mathrm{w}} \\ \underline{\theta} \\ \underline{\phi} \end{bmatrix} \tag{1.66}$$

where the transformation matrix is block-defined by appending to the 3x8 block introduced in Eqn. 1.64 a 3x3 zero block (the $\underline{\mathrm{w}}$ out-of-plane displacements have no influence on the in-plane strain components), and then the 3x8 block presented in Eqn. 1.65.

By separating the terms of the above matrix based on their order with respect to z, we finally have.

$$\underline{\epsilon}(\xi,\eta,z) = \left(\underline{\underline{\mathrm{B}}}_0(\xi,\eta) + \underline{\underline{\mathrm{B}}}_1(\xi,\eta)z\right)\underline{\mathrm{d}} \tag{1.67}$$

The out-of-plane shear strain components, as defined in Eqns. 1.33 and 1.34, become

$$\begin{bmatrix} \bar{\gamma}_{zx} \\ \bar{\gamma}_{yz} \end{bmatrix} = \underline{\underline{\nabla}}\,(\xi,\eta)\,\underline{\mathrm{w}} + \begin{bmatrix} 0 & +\underline{\underline{\mathrm{N}}}\,(\xi,\eta) \\ -\underline{\underline{\mathrm{N}}}\,(\xi,\eta) & 0 \end{bmatrix} \begin{bmatrix} \underline{\theta} \\ \underline{\varphi} \end{bmatrix}, \tag{1.68}$$

and thus, by employing a notation consistent with 1.67,

$$\begin{bmatrix} \bar{\gamma}_{zx} \\ \bar{\gamma}_{yz} \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{\nabla}}\,(\xi,\eta) & \begin{matrix} 0 \\ -\underline{\underline{\mathrm{N}}}\,(\xi,\eta) \end{matrix} & \begin{matrix} \underline{\underline{\mathrm{N}}}\,(\xi,\eta) \\ 0 \end{matrix} \end{bmatrix}}_{\underline{\underline{\mathbf{B}}}_{\bar{\gamma}}(\xi,\eta)} \underline{\mathrm{d}} \tag{1.69}$$

where the transformation matrix that derives the out-of-plane, interlaminar strains from the nodal degrees of freedom vector is constituted by five 2×4 blocks.

Bibliography

[1] C. Hua, “An inverse transformation for quadrilateral isoparametric elements: analysis and application,” *Finite elements in analysis and design*, vol. 7, no. 2, pp. 159–166, 1990.