Chapter 9 Torsion of Thin-Walled Tubes

Summary of Saint-Venant Torsion Theory

Warping function, Ψ

shear stress $\tau_{xy} = G\beta(\frac{\partial\Psi}{\partial y} - z), \quad \tau_{xz} = G\beta(\frac{\partial\Psi}{\partial z} + y)$

compatibility relationship automatic

equilibrium equations $\nabla^2\Psi = 0$

boundary conditions $\tau_{xy}\frac{dz}{ds} - \tau_{xz}\frac{dy}{ds} = 0$

$$\frac{\partial\Psi}{\partial y}\frac{dz}{ds} - \frac{\partial\Psi}{\partial z}\frac{dy}{ds} = \frac{1}{2}\frac{d}{ds}(y^2 + z^2)$$

torque $T = \iint_A (\tau_{xz}\, y - \tau_{xy}\, z)\, dA$

$$T = G\beta\iint_A (\frac{\partial\Psi}{\partial z} y - \frac{\partial\Psi}{\partial y} z)\, dA + G\beta J$$

Prandtl stress function, ϕ

shear stress $\tau_{xy} = \frac{\partial\phi}{\partial z}, \quad \tau_{xz} = -\frac{\partial\phi}{\partial y}$

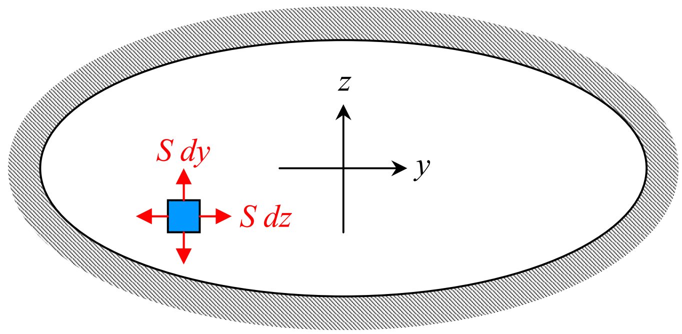
compatibility relationship $\nabla^2\phi = -2G\beta$

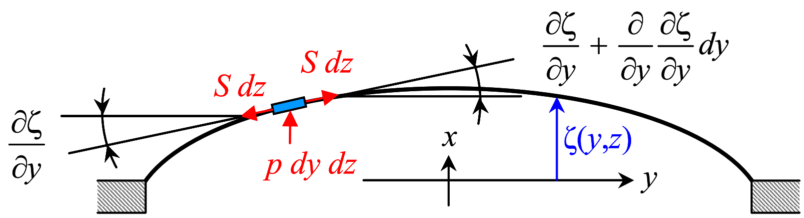
equilibrium equations automatic

boundary conditions $d\phi = 0$

torque $T = 2\iint_A \phi\, dA$

Membrane Analogy





$$dF_x = -S\,dz\sin\left(\frac{\partial \zeta}{\partial y}\right) + S\,dz\sin\left(\frac{\partial \zeta}{\partial y} + \frac{\partial}{\partial y}\frac{\partial \zeta}{\partial y}dy\right)$$

$$-S\,dy\sin\left(\frac{\partial \zeta}{\partial z}\right) + S\,dy\sin\left(\frac{\partial \zeta}{\partial z} + \frac{\partial}{\partial z}\frac{\partial \zeta}{\partial z}dz\right) + p\,dy\,dz = 0$$

$$S\,dz\frac{\partial^2 \zeta}{\partial y^2}dy + S\,dy\frac{\partial^2 \zeta}{\partial z^2}dz + p\,dy\,dz = 0$$

$$\boxed{\nabla^2 \zeta = \frac{\partial^2 \zeta}{\partial y^2} + \frac{\partial^2 \zeta}{\partial z^2} = -\frac{p}{S}}$$

compatibility relationship:

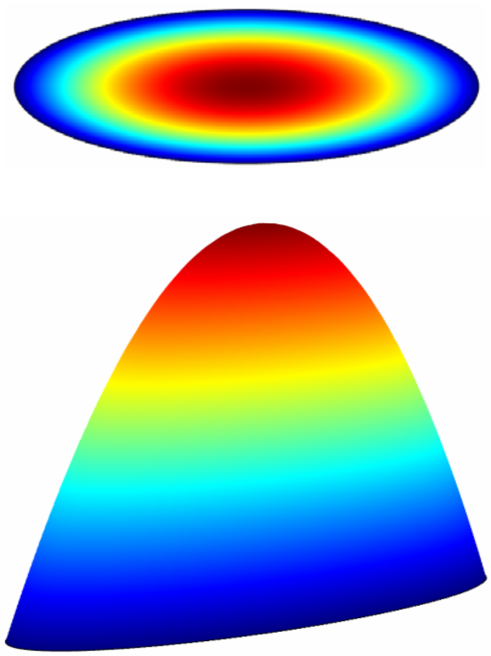
$$\nabla^2 \phi = -2G\beta$$

boundary condition:

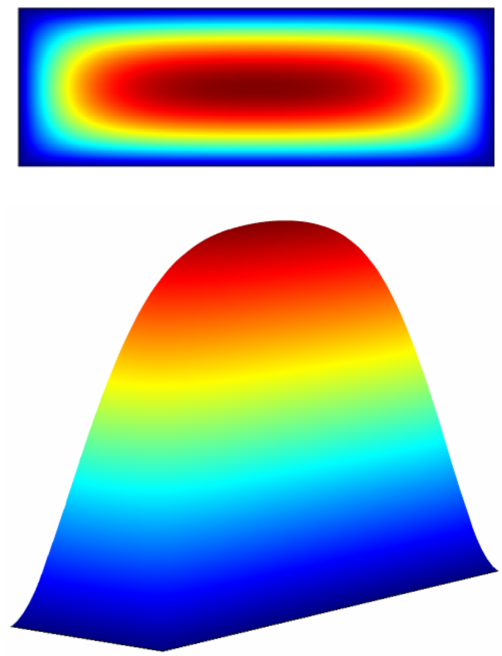
$$d\phi = 0$$

Examples of Solid Cross Sections

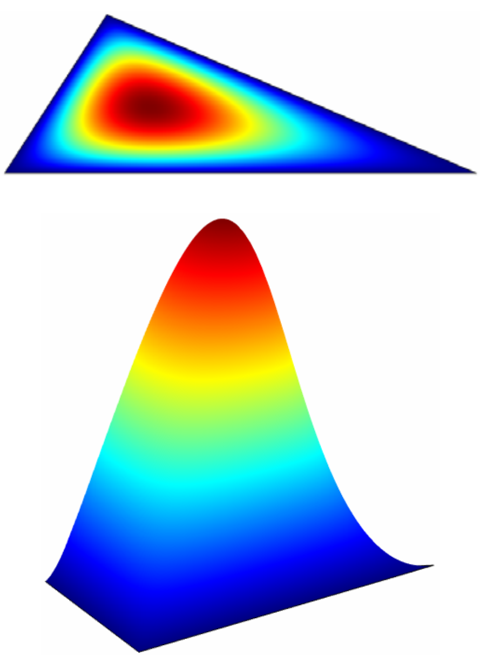
elliptical cross section



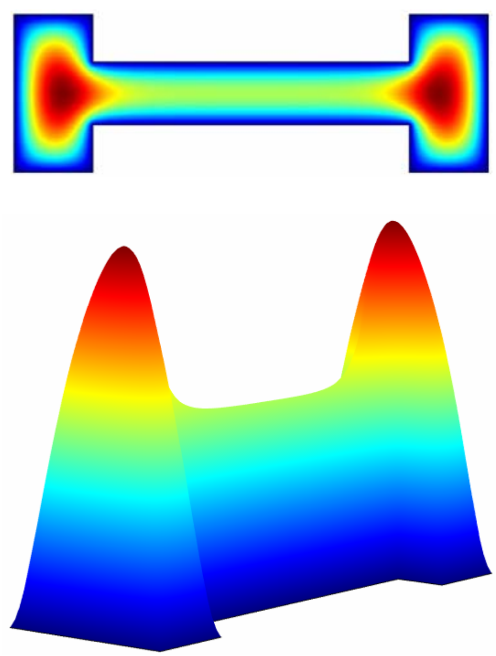
rectangular cross section



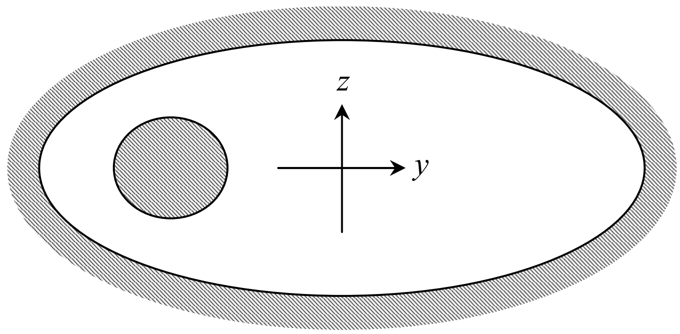
triangular cross section

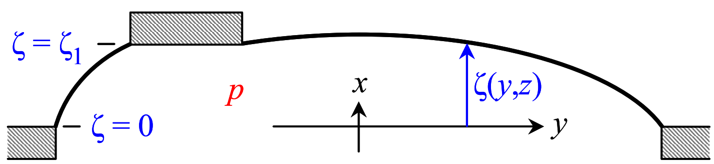


I-beam cross section

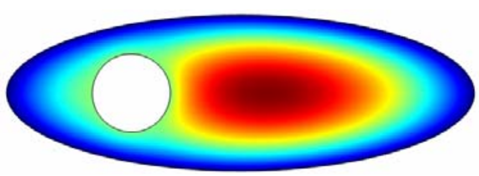


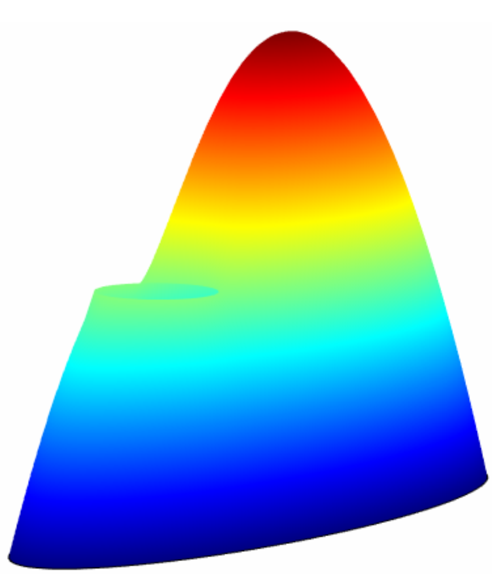
Discontinuous Boundary



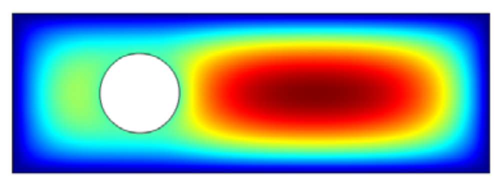


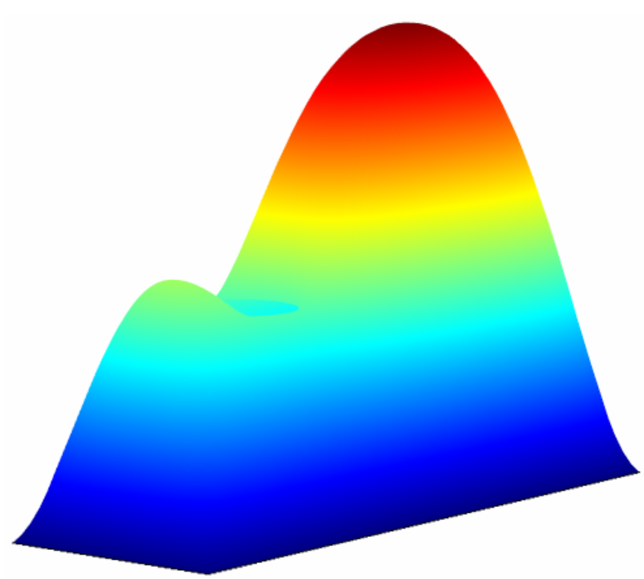
elliptical cross section with hole





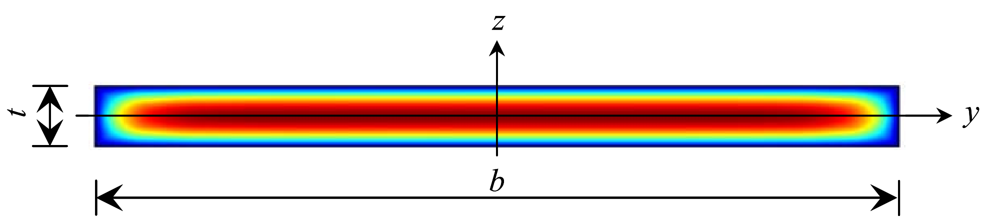
rectangular cross section with hole





Narrow Rectangular Cross Section

$$b >> t$$



equilibrium in x-direction

$$dF_x = S\,dy\,\frac{\partial^2\zeta}{\partial z^2}dz + p\,dy\,dz = 0$$

$$\frac{\partial^2\zeta}{\partial z^2} = -\frac{p}{S}$$

$$\frac{\partial\zeta}{\partial z} = -\frac{p}{S}z + 0$$

$$\zeta = -\frac{p}{2S}z^2 + C$$

$$\zeta(z=\pm\frac{t}{2}) = -\frac{p}{8S}t^2 + C = 0$$

$$C = \frac{p}{8S}t^2$$

$$\zeta = \frac{p}{2S}\left(\frac{t^2}{4} - z^2\right)$$

$$\boxed{\phi = G\beta\left(\frac{t^2}{4} - z^2\right)}$$

torque

$$T = 2\iint_A \phi\, dA = 2G\beta b \int_{z=-t/2}^{t/2} \left(\frac{t^2}{4} - z^2 \right) dz$$

$$T = 2G\beta b \left[\frac{t^2 z}{4} - \frac{z^3}{3} \right]_{-t/2}^{t/2}$$

$$\boxed{T = \frac{1}{3} G\beta b t^3}$$

shear stress

$$\tau_{xy} = \frac{\partial\phi}{\partial z} = -2G\beta z\,, \qquad \tau_{xz} = -\frac{\partial\phi}{\partial y} \approx 0$$

$$\tau_{\text{max}} = G\beta t = \frac{3T}{bt^2}$$

torsional stiffness

$$K\ (= J_t) = \frac{T}{\beta G}$$

$$K = \frac{1}{3} b t^3$$

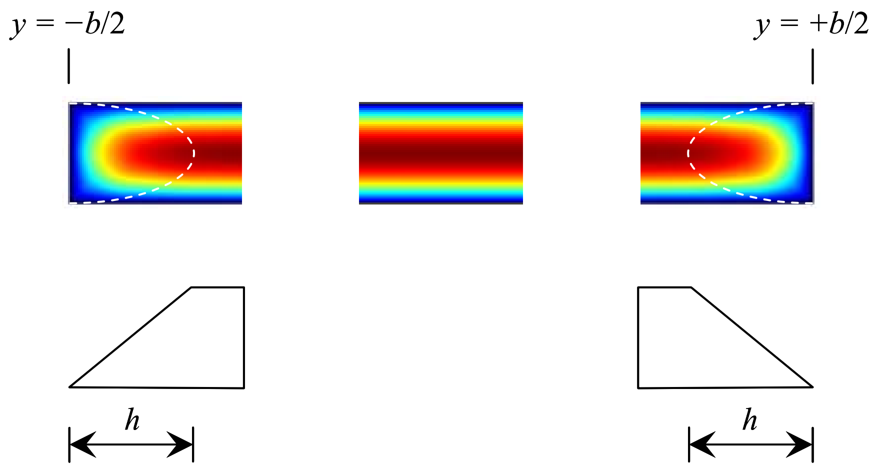
torque contributions

$$T = \iint_A (-\tau_{xy} z + \tau_{xz} y)\, dA = T_{\text{bulk}} + T_{\text{end}}$$

$$T_{\text{bulk}} = -\iint_A \tau_{xy} z\, dA = 2G\beta b \int_{z=-t/2}^{t/2} z^2\, dz$$

$$\boxed{T_{\text{bulk}} = \frac{1}{6} G\beta b t^3 = \frac{1}{2} T}$$

End Correction



1D approximation:

$$\phi = G\beta\left(\frac{t^2}{4} - z^2\right) \quad \text{if} \quad -\frac{b}{2} \le y \le \frac{b}{2} \quad \text{and} \quad \phi = 0 \quad \text{else}$$

$$\tau_{xz} = -\frac{\partial\phi}{\partial y} = 0 \quad \text{if} \quad -\frac{b}{2} < y < \frac{b}{2} \quad \text{and} \quad \frac{\partial\phi}{\partial y} = \pm\infty \quad \text{if} \quad y = \mp\frac{b}{2}$$

2D "tapered" approximation:

$$\phi = G\beta\left(\frac{t^2}{4} - z^2\right) \quad \text{if} \quad -\frac{b}{2} + h \le y \le \frac{b}{2} - h \quad \text{and} \quad \phi = G\beta\left(\frac{t^2}{4} - z^2\right)\frac{\frac{b}{2} - y}{h} \quad \text{else}$$

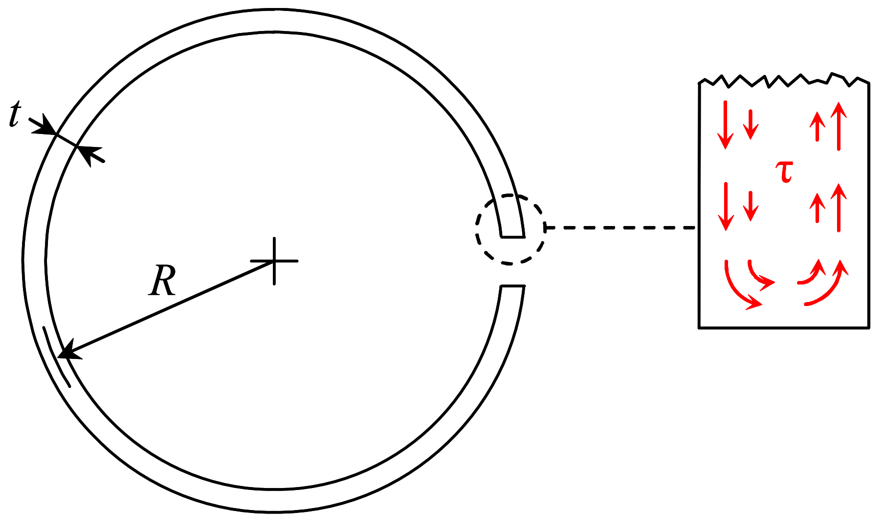
$$dT_{\text{end}} = \tau_{xz}\, y\, dA$$

$$T_{\text{end}} = \iint_A \tau_{xz}\, y\, dA \approx G\beta\left[\int_{z=-t/2}^{t/2}\left(\frac{t^2}{4} - z^2\right)dz\right]\left[\frac{b}{h}\int_{b/2-h}^{b/2}\left(\frac{b}{2} - y\right)dy\right]$$

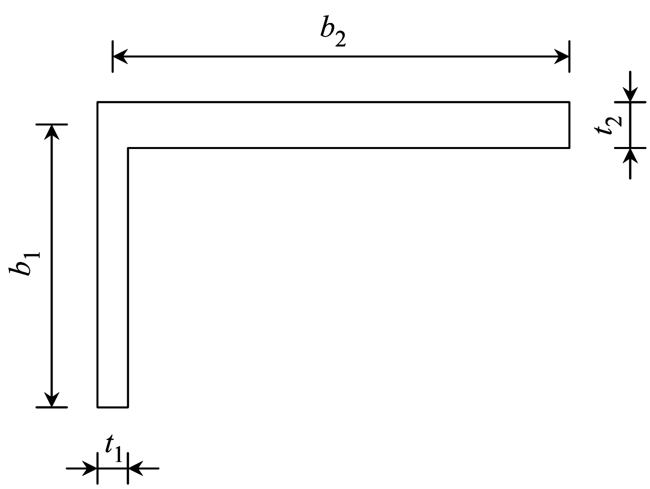
$$\boxed{T_{\text{end}} = \frac{1}{6}G\beta\, b\, t^3 = \frac{1}{2}T}$$

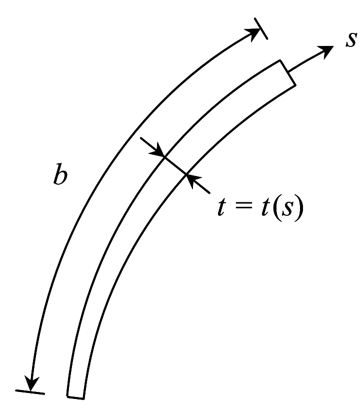
General Thin-Walled Shapes

$$K = \frac{T}{\beta G}$$



$$K = \frac{1}{3} b t^3 \le \frac{2\pi}{3} R t^3$$

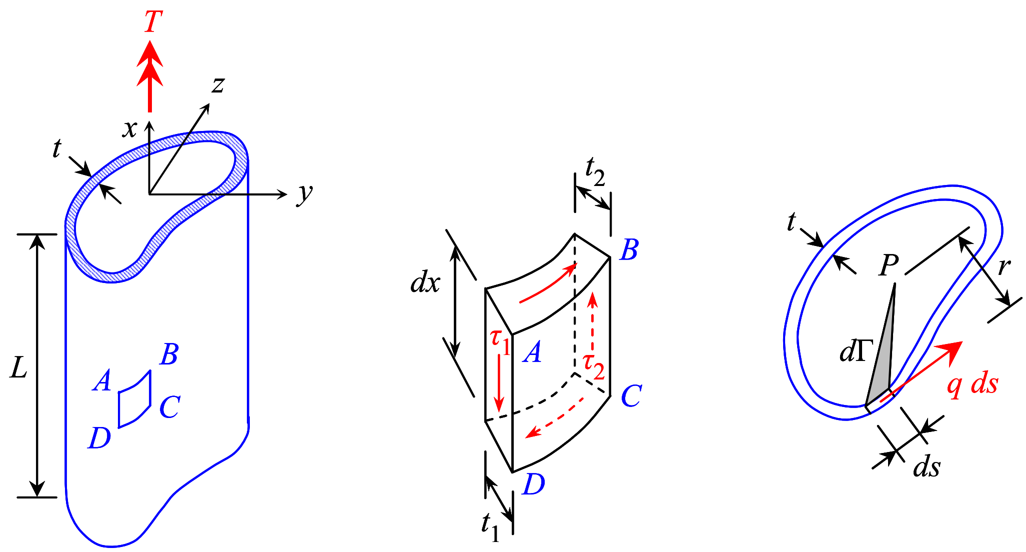




$$K = \frac{1}{3} \sum_{i=1}^{n} b_i t_i^3$$

$$K = \frac{1}{3} \int_0^b t^3(s)\, ds$$

Single-Cell Thin-Walled Tube



$$dF_x = -\tau_1 t_1 dx + \tau_2 t_2 dx = 0$$

constant shear flow q

$$q = \tau_1 t_1 = \tau_2 t_2$$

twisting moment T

$$dT = dF\, r = \tau t\, ds\, r = q\, ds\, r$$

$$d\Gamma = \frac{1}{2} ds\, r$$

$$dT = 2 q\, d\Gamma$$

$$\boxed{T = 2 q \Gamma}$$

$$q = \frac{T}{2\Gamma}$$

$$\tau \ (= \tau_{surf}) = \frac{T}{2\Gamma t}$$

Rate of Twist, Torsional Stiffness

total strain energy over length L

$$W = \frac{1}{2} T \beta L = U = \iiint_V U_0 \, dV = \oint \frac{\tau^2}{2G} L \, t \, ds$$

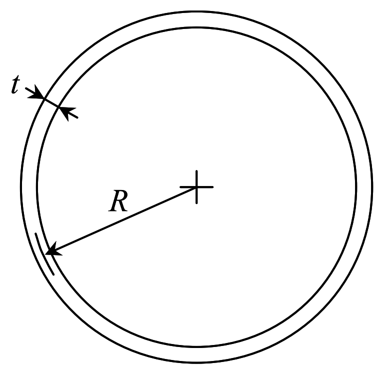
$$\frac{1}{2} T \beta L = \frac{T^2 L}{8 G \Gamma^2} \oint \frac{ds}{t}$$

Bredt's formula

$$\beta = \frac{T}{4 G \Gamma^2} \oint \frac{ds}{t}$$

$$\beta = \frac{q}{2 G \Gamma} \oint \frac{ds}{t}$$

Example: closed tube of circular cross section:

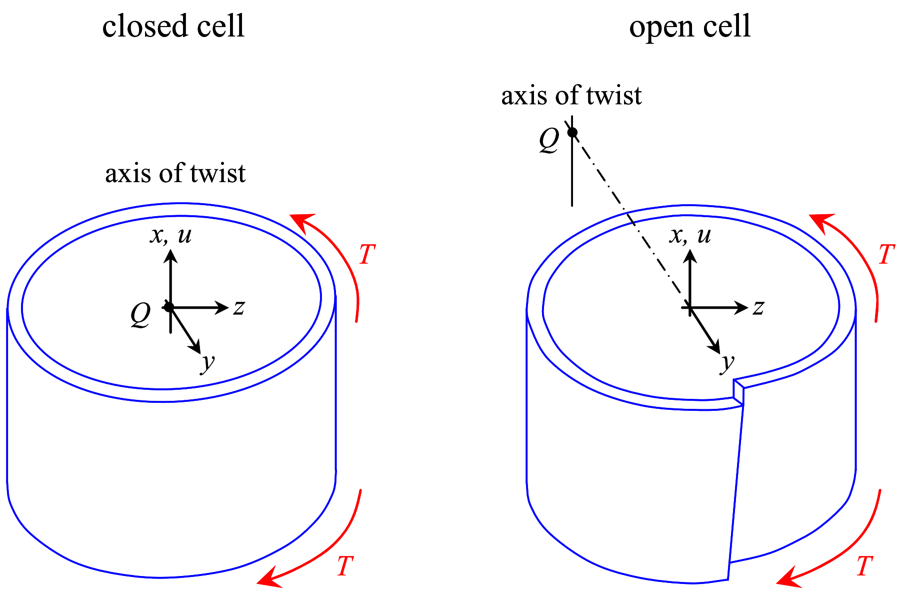


$$\beta^{closed} = \frac{T \ell}{4 G \Gamma^2 t}, \quad \ell = 2 \pi R, \quad \Gamma = \pi R^2$$

$$\beta^{closed} = \frac{T}{2 \pi G R^3 t}$$

$$\boxed{K^{closed} = 2 \pi R^3 t}$$

Warping Displacement



$$K^{\text{closed}} = 2\pi R^3 t$$

$$K^{\text{open}} = \frac{2\pi}{3} R t^3 << K^{\text{closed}}$$

$$\frac{K^{\text{closed}}}{K^{\text{open}}} = 3\frac{R^2}{t^2}$$

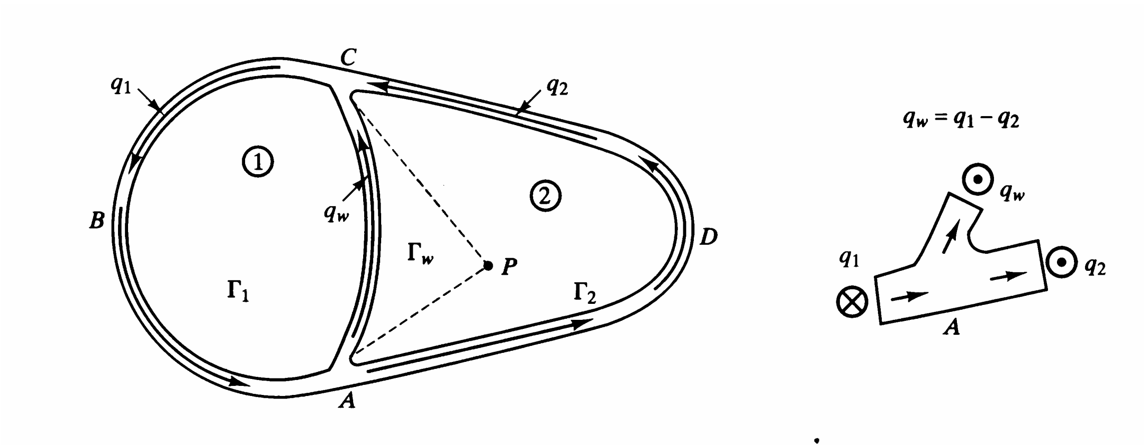
given torque T:

$$\tau^{\text{closed}} = \frac{T}{2\Gamma t} = \frac{T}{2\pi R^2 t}$$

$$\tau^{\text{open}} \;(= \tau^{\text{open}}_{\text{surf}}) = \frac{3T}{bt^2} = \frac{3T}{2\pi R t^2}$$

$$\frac{\tau^{\text{open}}}{\tau^{\text{closed}}} = 3\frac{R}{t}$$

Multicell Tubes



$$F_{axial} = -q_1 L + q_2 L + q_w L = 0$$

$$q_w = q_1 - q_2$$

$$T = \int_{CBA} q_1\, r\, ds + \int_{ADC} q_2\, r\, ds + \int_{AC} q_w\, r\, ds$$

$$T = 2(\Gamma_1 + \Gamma_w) q_1 + 2(\Gamma_2 - \Gamma_w)\, q_2 - 2\,\Gamma_w\, q_w$$

$$\boxed{T = 2\Gamma_1 q_1 + 2\Gamma_2 q_2}$$

For general n-cell cross section:

$$T = 2\sum_{i=1}^{n} \Gamma_i q_i$$

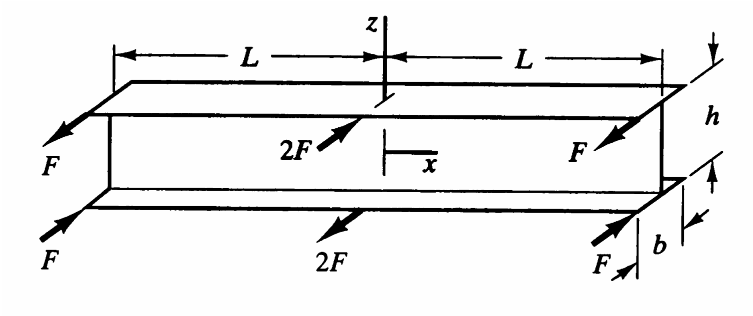
shear flow q_i in the ith loop:

$$2G\beta = \frac{1}{\Gamma_i} \oint \frac{q_i}{t_i}\, ds_i$$

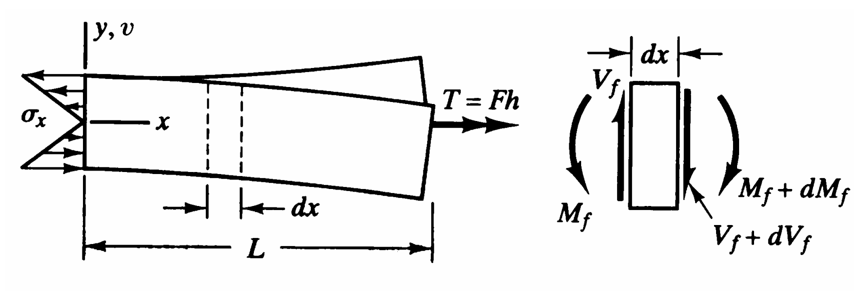
Restraint of Warping

Example: I Sections

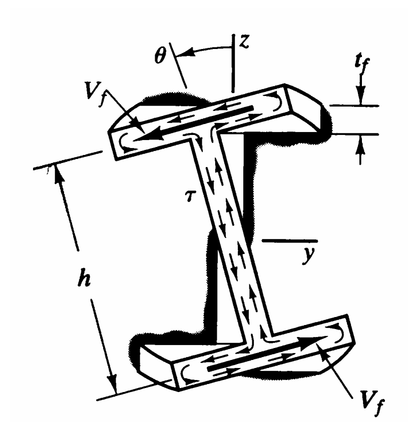
symmetric loading $u(x = 0, y, z) = 0$



top view of the upper flange with differential element



axial view



$\beta = \beta(x)$

bending of the upper flange:

$$M_f = -E I_f \frac{\partial^2 v}{\partial x^2} \qquad I_f = \frac{t_f b^3}{12}$$

$$v = -\frac{h}{2}\theta$$

$$M_f = \frac{1}{2} E I_f h \frac{\partial^2 \theta}{\partial x^2} = \frac{1}{2} E I_f h \frac{\partial \beta}{\partial x}$$

$$V_f = -\frac{\partial M_f}{\partial x} = -\frac{1}{2} E I_f h \frac{\partial^2 \beta}{\partial x^2}$$

torque due to shear force in the flanges:

$$T_f = V_f h$$

total torque:

$$T = V_f h + K G \beta = -\frac{1}{2} E I_f h^2 \frac{\partial^2 \beta}{\partial x^2} + K G \beta$$

$$J_\omega = I_f \frac{h^2}{2} = \frac{t_f b^3 h^2}{24}$$

(called the sectorial area of inertia)

$$T = -E J_\omega \frac{\partial^2 \beta}{\partial x^2} + K G \beta$$

$$\frac{\partial^2 \beta}{\partial x^2} - \frac{K G}{E J_\omega}\beta = -\frac{T}{E J_\omega}$$

$$\frac{\partial^2 \beta}{\partial x^2} - k^2 \beta = -\frac{k^2 T}{G K} \quad \text{where} \quad k^2 = \frac{K G}{E J_\omega} = \frac{K}{2(1+\nu) J_\omega}$$

$$\boxed{\beta(x) = C_1 \sinh kx + C_2 \cosh kx + \frac{T}{G K}}$$

boundary conditions:

$$\beta(x=0) = 0 \qquad \text{(no warping at the center)}$$

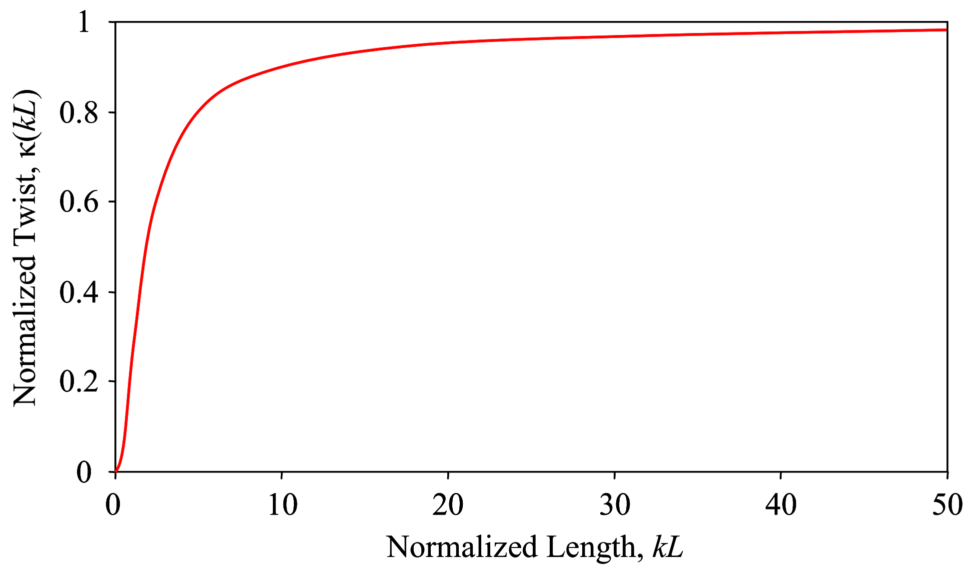
$$\left.\frac{\partial\beta}{\partial x}\right|_{x=L} = 0 \qquad \text{(no bending moment at the end)}$$

$$C_2 = -\frac{T}{GK} \qquad \text{and} \qquad C_1 = \frac{T}{GK}\tanh kL$$

$$\beta(x) = \frac{T}{GK}\left(\tanh kL \sinh kx - \cosh kx + 1\right)$$

$$\tau \ (=\tau_{\text{surf}}) = G\beta t$$

$$\theta(L) = \int_0^L \beta(x)\,dx = \frac{TL}{GK}\left(1 - \frac{\tanh kL}{kL}\right) = \frac{TL}{GK}\kappa(kL)$$



$$M_f(x) = \frac{EI_f hTk}{2GK}\left(\tanh kL \cosh kx - \sinh kx\right)$$

$$V_f(x) = -\frac{EI_f hTk^2}{2GK}\left(\tanh kL \sinh kx - \cosh kx\right)$$