0.1 Mass matrix for the finite element

0.1.1 Energy consistent formulation for the mass matrix

The Ω volume of material associated to a finite element is considered, along with the local, physical reference system (x, y, z), and its isoparametric counterpart that, for the quadrilateral plate element under scrutiny, is embodied by the (ξ, η, z) triad.

The *vector* shape function array

$$\underline{\underline{\mathrm{S}}}\,(\xi,\eta,z) = \begin{bmatrix} \dots & \tilde{u}_i(\xi,\eta,z) & \dots \\ \dots & \tilde{v}_i(\xi,\eta,z) & \dots \\ \dots & \tilde{w}_i(\xi,\eta,z) & \dots \end{bmatrix} \tag{1}$$

is defined based on the elementary motions $\underline{\tilde{\mathrm{u}}}_i \equiv [\tilde{u}_i, \tilde{v}_i, \tilde{w}_i]^\top$ induced to the element material points by imposing a unit value to the i-th degree of freedom d_i, while keeping the others fixed.

The displacement field is then defined as a linear combination of the elementary motions above, where the $\underline{\mathrm{d}}$ element Degree of Freedom (DOF)s serve as coefficients, namely

$$\underline{\mathrm{u}}\,(\xi,\eta,z) = \underline{\underline{\mathrm{S}}}\,(\xi,\eta,z)\,\underline{\mathrm{d}}\,. \tag{2}$$

Deriving with respect to time the equation above, the velocity field

$$\underline{\dot{\mathrm{u}}}\,(\xi,\eta,z) = \underline{\underline{\mathrm{S}}}\,(\xi,\eta,z)\,\underline{\dot{\mathrm{d}}} \tag{3}$$

is obtained as a function of the first variation in time of element DOFs. Expression 3 is simplified by the constant-in-time nature of $\underline{\underline{\mathrm{S}}}$.

The kinetic energy contribution associated to the deformable element material points may be integrated, thus obtaining

$$E_{\mathrm{kin}} = \frac{1}{2}\iiint_\Omega \underline{\dot{\mathrm{u}}}^{\top}\,\underline{\dot{\mathrm{u}}}\,\rho d\Omega \tag{4}$$

where ρ is the material mass density, that may vary across the domain. By substituting the velocity field definition of Eq. 3 we obtain

$$E_{\mathrm{kin}} = \frac{1}{2}\iiint_\Omega \left[\underline{\underline{\mathrm{S}}}\,\underline{\dot{\mathrm{d}}}\right]^{\top}\left[\underline{\underline{\mathrm{S}}}\,\underline{\dot{\mathrm{d}}}\right]\rho d\Omega, \tag{5}$$

and finally

$$E_{\text{kin}} = \frac{1}{2}\,\underline{\dot{\mathrm{d}}}^{\top}\left[\iiint_{\Omega}\underline{\underline{\mathrm{S}}}^{\top}\,\underline{\underline{\mathrm{S}}}\,\rho d\Omega\right]\,\underline{\dot{\mathrm{d}}} = \frac{1}{2}\,\underline{\dot{\mathrm{d}}}^{\top}\,\underline{\underline{\mathrm{M}}}\,\underline{\dot{\mathrm{d}}}\,. \tag{6}$$

The integral term that defines the $\underline{\underline{\mathrm{M}}}$ *mass* matrix is evaluated by resorting to the same quadrature technique introduced for its stiffness counterpart.

The actual nature of the mass matrix terms varies based on the type of the DOFs that are associated to the term row and column; in particular, the diagonal terms that are related to displacements and rotations are dimensionally consistent with a mass and a moment of inertia, respectively.

The mass matrix quantifies the inertial response of the finite element; according to its definition

$$\underline{\underline{\mathrm{M}}} = \iiint_{\Omega}\underline{\underline{\mathrm{S}}}^{\top}\,\underline{\underline{\mathrm{S}}}\,\rho d\Omega, \tag{7}$$

it is merely a function of the material density, and of the kinematic laws that constrain the motion of the material particles within the element.

If a set of external (generalized) forces $\underline{\mathrm{F}}$ is applied to the element DOFs in the fictitious absence of elastic reactions, a purely inertial response is expected. The $\underline{\dot{\mathrm{d}}}$ vector defines the instantaneous first derivative in time of the DOFs (i.e. nodal translational and rotational velocities); the instantaneous power supplied by the external forces is then evaluated as $\underline{\dot{\mathrm{d}}}^{\top}\,\underline{\mathrm{F}}$, that induces an equal time derivative of the kinetic energy, quantified as [1]

$$\begin{aligned}\underline{\dot{\mathrm{d}}}^{\top}\,\underline{\mathrm{F}} = \frac{dE_{\text{kin}}}{dt} =& \frac{d}{dt}\left(\frac{1}{2}\,\underline{\dot{\mathrm{d}}}^{\top}\,\underline{\underline{\mathrm{M}}}\,\underline{\dot{\mathrm{d}}}\right)\\ =& \frac{1}{2}\left(\underline{\ddot{\mathrm{d}}}^{\top}\,\underline{\underline{\mathrm{M}}}\,\underline{\dot{\mathrm{d}}} + \underline{\dot{\mathrm{d}}}^{\top}\,\underline{\underline{\mathrm{M}}}\,\underline{\ddot{\mathrm{d}}}\right)\\ =& \underline{\dot{\mathrm{d}}}^{\top}\,\underline{\underline{\mathrm{M}}}\,\underline{\ddot{\mathrm{d}}}\,.\end{aligned}$$

[1] The symmetric matrix characterizing property

$$\underline{\mathrm{x}}^{\top}\,\underline{\underline{\mathrm{A}}}\,\underline{\mathrm{y}} = \underline{\mathrm{y}}^{\top}\,\underline{\underline{\mathrm{A}}}\,\underline{\mathrm{x}} \quad \forall\underline{\mathrm{x}},\underline{\mathrm{y}} \in \mathbb{R}^{n}$$

is used in deriving the last passage.

Due to the general nature of $\underline{\dot{d}}$, equality

$$\underline{F} = \underline{\underline{M}}\, \underline{\ddot{d}} \tag{8}$$

is implied, which points out the mass matrix role in transforming the DOF vector second derivative in time (i.e. nodal translational and rotational accelerations) into the generalized force components that are to be applied in order to sustain such variation of motion.

0.1.2 Lumped mass matrix formulation

In a few applications, a diagonal form for the mass matrix is preferred at the expense of a) a strict adherence to energy consistency, and b) some arbitrariness in its definition.

The finite element volume is ideally partitioned into a set of influence domains, one each node. In the case of the four-noded quadrilateral, material points whose ξ, η isoparametric coordinates fall within the first, second, third and fourth quadrant are associated to nodes n3, n4, n1 and n2, respectively; those distributed masses are then ideally accumulated at the associated node.

A group of four concentrated nodal masses is thus defined, whose motion is defined based on single translational DOFs, and not on the plurality of weighted contributions that induces the nonzero, nondiagonal terms at the consistent mass matrix.

This undue material accumulation at the element periphery produces a spurious increase of the moment of inertia, condition, this, that may only be worsened if (positive) rotational inertias are introduced at nodes.

Those nodal rotational inertias are however required in associating a bounded angular acceleration to unbalanced nodal torques; solution methods based on the mass matrix inversion, e.g. explicit dynamic procedures, are precluded otherwise. Since there is no consensus on the quantification those inertial terms, the reader is addressed to specialized literature.

0.2 External forces

Energetically consistent external actions may be applied at the nodal DOFs, that may be interpreted as *concentrated* forces and moments;

their physical rationalization outside the discretized structure framework – and in particular back to the underlying elastic continua theory – is far from being trivial.

Surface tractions and volumetric loads are instead naturally tied with the continuum formulation, and are usually employed in formalizing the load condition of structural components.

The present paragraph derives the equivalent nodal representation of distributed actions acting on the domain of a single finite element; the inverse relation provides a finite, distributed traction counterpart to concentrated actions applied at the nodes of a discretized FE model.

The $\underline{\underline{\mathrm{S}}}$ set of elementary deformation modes that is introduced in the context of the element mass matrix derivation, see Eqn. 1, is employed to define a virtual displacement field within the element domain based on the virtual variation $\delta\,\underline{\mathrm{d}}$ of its nodal DOFs values, i.e.

$$\delta\,\underline{\mathrm{u}}\,(\xi,\eta,z) = \underline{\underline{\mathrm{S}}}\,(\xi,\eta,z)\delta\,\underline{\mathrm{d}}\,, \tag{9}$$

see also Eq. 2.

A volumetric external load is considered, whose components $\underline{\mathrm{q}} = [q_x, q_y, q_z]$ are consistent with the $\underline{\underline{\mathrm{S}}}$ matrix reference system, i.e. the local to the element, physical Cxyz one. If external load components are defined in the context of a global reference system, straightforward reference frame transformations are to be applied.

The virtual work performed by those distributed actions is first integrated along the element domain, and then equalled to its nodal counterpart $\delta\,\underline{\mathrm{d}}^{\top}\,\underline{\mathrm{F}}$, thus leading to

$$\begin{aligned}
\delta\,\underline{\mathrm{d}}^{\top}\,\underline{\mathrm{F}} &= \iiint_{\Omega} (\delta\,\underline{\mathrm{u}})^{\top}\,\underline{\mathrm{q}}\,d\Omega \\
&= \iiint_{\Omega} \left(\underline{\underline{\mathrm{S}}}\,\delta\,\underline{\mathrm{d}}\right)^{\top}\,\underline{\mathrm{q}}\,d\Omega \\
&= \delta\,\underline{\mathrm{d}}^{\top} \iiint_{\Omega} \underline{\underline{\mathrm{S}}}^{\top}\,\underline{\mathrm{q}}\,d\Omega,
\end{aligned}$$

and finally to

$$\underline{\mathrm{F}} = \iiint_{\Omega} \underline{\underline{\mathrm{S}}}^{\top}\,\underline{\mathrm{q}}\,d\Omega \tag{10}$$

due to the general nature of $\delta\,\underline{\mathrm{d}}$.

The quadrature along the domain is performed according to the methods introduced for deriving the element stiffness matrix. If a surface or an edge load are supplied in place of the volumetric load vector $\underline{q}$, equation 10 integral may be adapted to span each loaded element face, or edge.

In the case of low order isoparametric elements – e.g. the four-noded quadrilateral shell element, an alternative, simplified procedure for the consolidation of the distributed loads into nodal forces becomes viable. According to such procedure, the element domain is partitioned into influence volumes, one each node; the external load contributions are then accumulated within each partition, and the resultant force vector is applied to the associated node.

By moving such resultant force from the distribution Center of Gravity (COG) to the corner node, momentum balance is naively disregarded; the induced error however decreases with the load field variance across the element, and hence with the element size. Such error vanishes for uniform distributed loads.

In the presence of a better established, work consistent counterpart, such simplified procedure is mostly employed to set a rule-of-thumb equivalence between distributed and nodal loads; in particular, the stress-singular nature of a set of nodal loads may be easily pointed out if it is observed that a finite load resultant is applied to influence areas that cumulatively vanish with vanishing element size.