0.1 Shear stresses due to the St. Venant torsion

The classical solution for the rectilinear beam subject to uniform torsion predicts a displacement field that is composed by the superposition of a) a rigid, in-plane[1] cross section rotation about the shear centre, named twist, whose axial rate is uniform, and b) an out-of-plane *warping* displacement that is uniform in the axial direction, whereas it varies within the section; such warping displacement is zero in the case of axisymmetric sections only (e.g. solid and hollow circular cross sections).

Due to the rigid nature of the in-plane displacements, the in-plane strain components ϵ_x, ϵ_y, ϵ_{xy} are zero; the in-plane stress components σ_x, σ_y, τ_{xy}, and the normal stress σ_z are also zero if z is a direction of orthotropy for the material – as it is assumed in the following. The motion is internally restricted only due to the nonzero out-of-plane shear stresses τ_{yz} and τ_{zx}, that develop as an elastic reaction to the associated strain components.

A more in-depth treatise of the topic involves the solution of an plane, inhomogeneous Laplace partial differential equation with essential conditions imposed at the cross section boundary, which is beyond the scope of the present contribution.

However, in the case of open- and closed- section, thin walled beams, simplified solutions are available based on the assumptions that a) the out-of-plane shear stresses are locally aligned to the wall midsurface - i.e. $\tau_{zr} = 0$ leaving τ_{zs} as the only nonzero stress component[2], and b) the residual τ_{zs} shear component is either constant by moving through the wall thickness (closed section case), or it linearly varies with the through-thickness coordinate r.

0.1.1 Solid section beam

TODO.

[1] the rotation vector is actually normal to the cross sectional plane; the *in-plane* motion characterization refers to the associated displacement field.

[2] Here, the notation introduced in paragraph XXX for the thin walled section is employed.

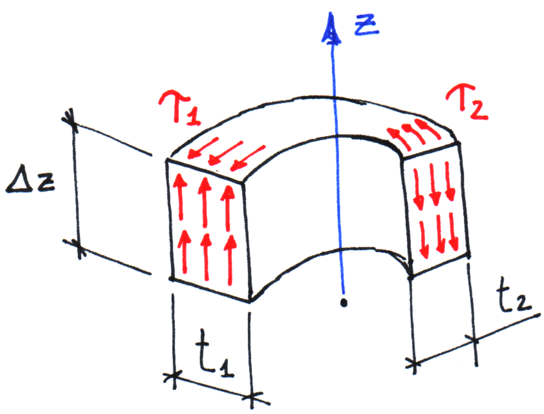

Figure 1: Axial equilibrium for a portion of profile wall, in the case of a closed, thin-walled profile subject to torsion.

0.1.2 Closed section, single-celled thin walled beam

The τ_{sz} component is assumed uniform along the wall thickness, or, equivalently, its deviation from the average value is neglected in calculations.

In the case the material is non-uniform across the thickness, the γ_{sz} shear strain is assumed uniform, whereas the τ_{sz} varies with the varying G_{sz} shear modulus.

In the absence of σ_z, the axial equilibrium of a portion of beam segment dictates that the shear flow $t\tau$ remains constant along the wall, i.e.

$$t_1\tau_1 = t_2\tau_2$$

as depicted in Figure 1.

By skipping some further interesting observations (TODO) we may just introduce the Bredt formula for the cross-section torsional stiffness

$$K_t = \frac{4A^2}{\oint \frac{1}{t} dl} \tag{1}$$

which is valid for single-celled, closed thin wall sections.

The peak stress is located at thinnest point along the wall, and equals

$$\tau_{\max} = \frac{M_t}{2t_{\min}A} \tag{2}$$

.

0.1.3 Closed section, multi-celled thin walled beam

TODO. However, a lower bound for the stiffness of the multi-celled thin walled beam may be obtained by fictitiosly severing the inner walls, thus obtaining a single cell defined by the outer wall alone.

An upper bound for the stiffness is obtained by assuming each shared inner wall as shear-rigid, and then by summing the stiffnesses of each elementary closed loop, as they constituted independent profiles. The shear-rigid nature of the inner walls is enforced by neglecting their contribution to the circuital integral at the Bredt formula denominator.

0.1.4 Open section, thin walled beam

The shear strain component γ_{zs} is assumed linearly varying across the thickness; if the G_{sz} shear modulus is assumed uniform, such linear variation characterizes the τ_{zs} stress components too.

The average value along the thickness of the τ_{zs} stress component is zero, as zero is the shear flow as defined in the previous paragraph.

For thin enough open sections of uniform and isotropic material we have

$$K_T \approx \frac{1}{3}\int_0^l t^3(s)ds \tag{3}$$

If the thin-walled cross section may be described as a sequence of constant thickness wall segments, the simplified formula

$$K_T \approx \frac{1}{3}\sum_i l_i t_i^3 \tag{4}$$

is obtained where t_i and l_i are respectively the length and the thickness of each segment.

The peak value for the τ_{zs} stress component is observed in correspondence to thickest wall section point and it equates

$$\tau_{\max} = \frac{M_t t_{\max}}{K_T} \tag{5}$$

By applying the reported formulas to a rectangular section whose span length is ten times the wall thickness, the torsional stiffness is overestimated by slightly less than 7%; a similar relative error is reported in terms of shear stress underestimation.