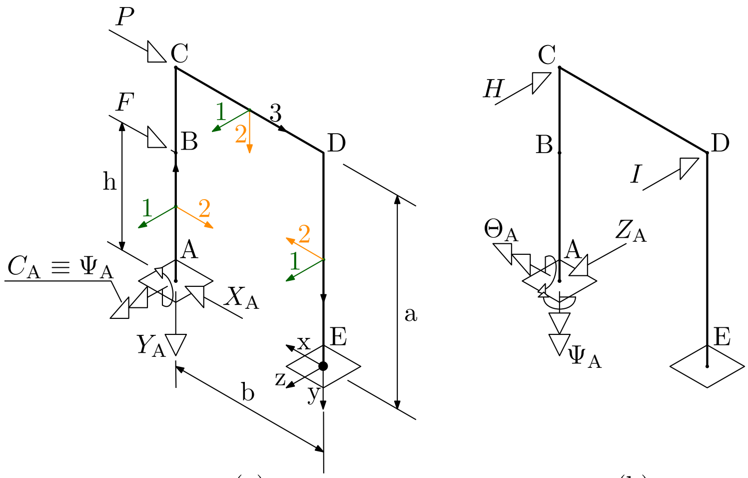


(a) (b)

Figure 1: A rollbar-like frame; Figures a) and b) collect the considered in-plane and out-of-plane actions, respectively, which are split for added readability.

0.1 A semi-worked example: a rollbar-like frame

Let's considered the plane frame structure depicted in Fig. 1, representing a simplified rollbar; a thin-walled, circular steel profile is employed for both the upright and the cross members, whose median diameter and thickness are d and t, respectively [1].

A global (E, xyz) reference system is employed[2] to represent the frame nodal coordinates, if required, and the constraint reaction components.

A local reference system $(G, 123)$ is set along the beam segments, whose third axis follows the beam branch orientation, and whose first axis is everywhere aligned with the global z direction.

The rollbar frame is clamped at both the A and E ends, and it is

[1]The present treatise is applicable to a generic material and cross section, provided that symmetry holds with respect to the plane the frame lies on; such further condition may be overcome coupling terms are considered between the otherwise uncoupled in-plane to out-of-plane problems.

[2]sorry for its unusual orientation, it has been inherited from some legacy lecture notes of mine.

loaded i) by a lateral force P, and ii) by a transverse force H, both applied at node C.

Expected results of the analysis are i) the internal action components at each frame node, and wherever they are maximal ii) the constraint reactions at the A and E clamps, iii) the lateral, inward[3] deflections $\tilde{u}_\mathrm{B}, \tilde{u}_\mathrm{C}$ at B and C, respectively, and iv) the transverse[4] deflections $\tilde{w}_\mathrm{C}, \tilde{w}_\mathrm{D}$ at both C and D.

The second Castigliano theorem is resorted to for deflection calculation, thus requiring the application of auxiliary, fictitious external forces F and I, that may perform work with the monitored deflection, if not already set (see the H force).

The structure is six times statically indeterminate; the clamp at A is removed, and the associated six components of constraint reaction, see Fig. 1, are set as further, parametrically defined external loads; a statically determinate *principal* stucture is hence obtained, which preserves the clamp at E as its only connection to ground.

The actual value of those parametrically defined loads is obtained by imposing a null deflection along each of the six d.o.f.s at node A, and thus casting a linear (due to the assumed structure behaviour) system of six equations in the aforementioned six unknown parameters. Again, the second Castigliano theorem is employed in evaluating the node A generalized displacements.

The expression of the structure internal strain energy, namely

$$U\left(P, F, X_\mathrm{A}, Y_\mathrm{A}, \Psi_\mathrm{A}, H, I, Z_\mathrm{A}, \Theta_\mathrm{A}, \Phi_\mathrm{A}\right)$$

is obtained as a function of the applied loads through the integration along the structure beam branches of the lineic strain energy density, which in turn depends on the pointwise value of the internal action components, see Eq. **??**.

Due to the symmetric nature of the structure under scrutiny, and to is assumed linear behavior, the overall problem may be partitioned into two uncoupled symmetric (or *in-plane*) and skew-symmetric(or *out-of-plane*) subproblems, that might be solved separately.

In order to streamline the treatise, the contribution alone is considered of the moment kind of stress resultants, thus neglecting the profile

[3] i.e. counter-oriented with respect to the x global axis
[4] i.e., oriented along the negative global z direction

compliance with respect to axial and shear internal actions; such customary approximation - consistent with an *inextensible* Euler beam model - is justified by the supposed profile slenderness.

Figure 2 collects the contribution of each the in-plane external actions to the $M1$ bending moment, plotted along the beam flank in tension. Such bending moment diagrams are obtained by considering the equilibrium of the portion of principal structure that spans from the A free end to each section which in turn is under scrutiny; please try to derive those diagrams on your own, since they *might hide some errors.*

We also notice that, consistently with the local axis orientation, $M1$ is assumed positive if it stretches the profile fibers that are inner with respect to the frame.

Similarly, Fig. 3 collects the M_2 bending moment component, assumed positive if it stretches the fibers on the "back" of the frame (i.e. the cross section points whose z or 1 coordinates are the most negative), along with the M_{t} torsional moment, whose sign is explicitly reported. Again, please derive them independently, since *some mistakes might be present.*

We observe that all the diagrams are branchwise linear, due to the piecewise straight centroidal segment nature, and the absence of distributed actions. In such condition, a generic M moment components may be conveniently expresses as

$$M(s) = M_0\, f\left(\frac{s}{l}\right) + M_l\, g\left(\frac{s}{l}\right), \qquad f(\xi) = 1 - \xi, \qquad g(\xi) = \xi$$

where $s \in [0, l]$ is a dimensional abscissa which spans through the l extension of each oriented segment, M_0 and M_l are the moment values at the extremities, and $\{f, g\}$ are two weight function whose aim is to linearly interpolate the moment extremal values along the beam segment interior.

For each segment, the associated strain energy is evaluated as

$$U = \int_0^l \underbrace{\frac{M_1^2(s)}{2EJ}}_{\text{symm.}} + \underbrace{\frac{M_2^2(s)}{2EJ} + \frac{M_{\mathrm{t}}^2(s)}{2GK_{\mathrm{t}}}}_{\text{skew−symm.}}\, ds, \tag{1}$$

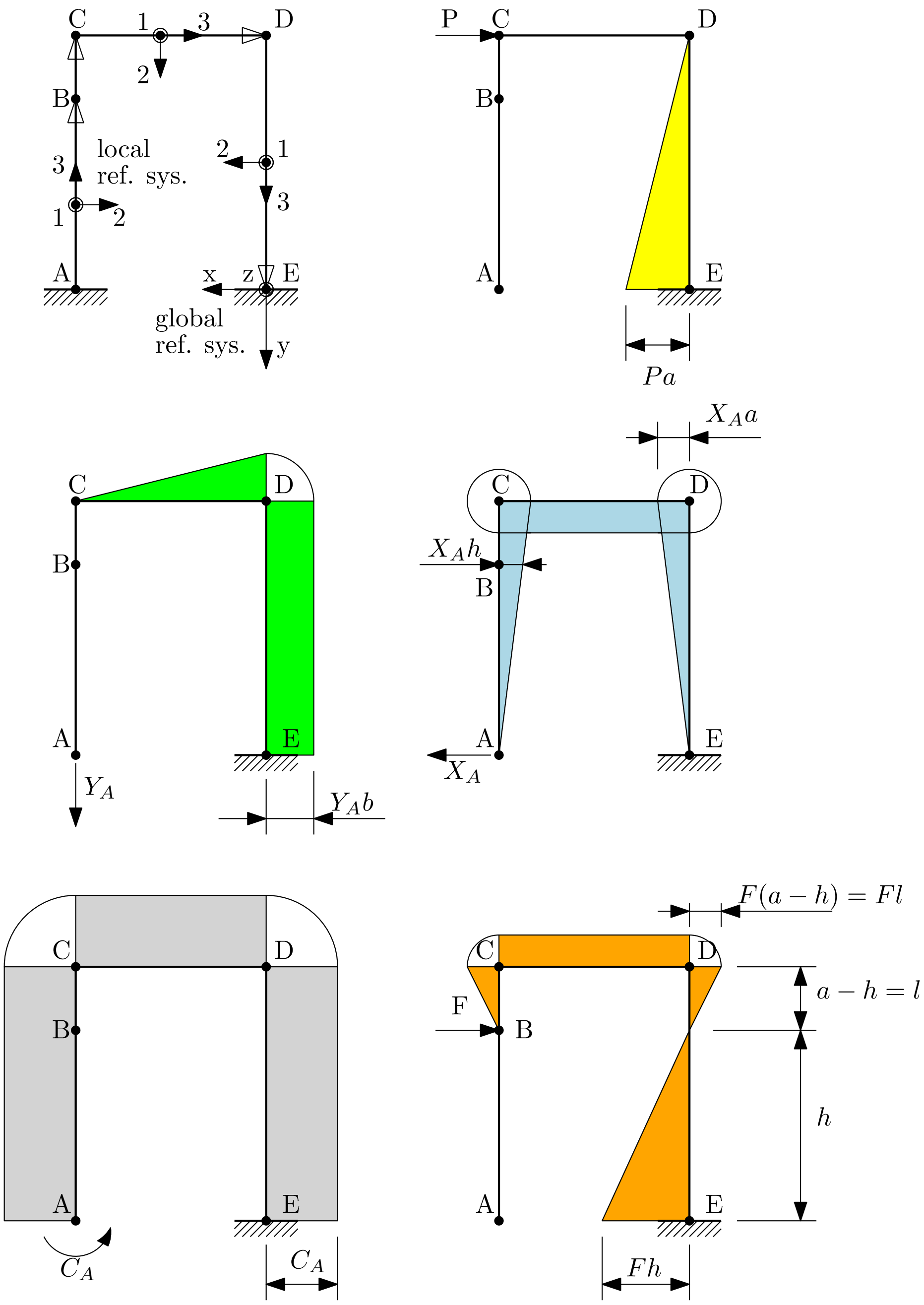
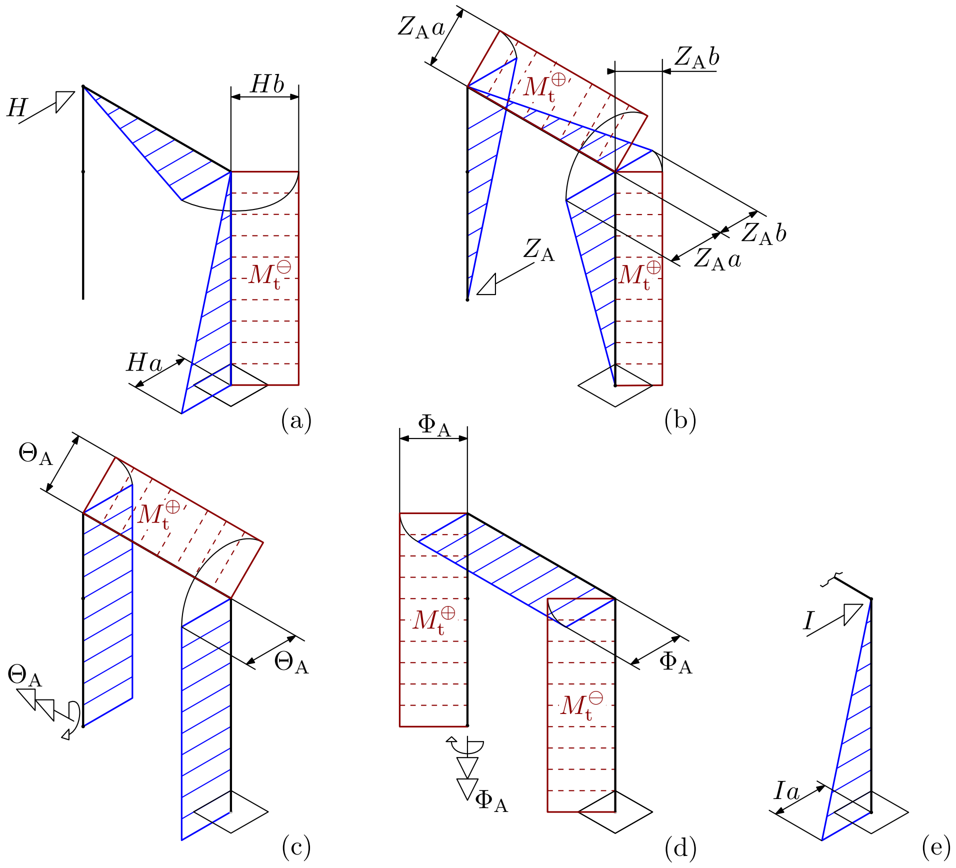

C
1
3
D
2
B
local
ref. sys.
2
1
3
3
1
2
A
x
z
E
global
ref. sys.
y
P
C
D
B
A
E
Pa
$X_A a$
C
D
B
A
E
Y_A
$Y_A b$
C
D
$X_A h$
B
A
E
X_A
C
D
B
A
E
C_A
C_A
$F(a-h) = Fl$
C
D
$a - h = l$
F
B
h
A
E
Fh

Figure 2: XXX



H
Hb
$M_t^{\ominus}$
Ha
(a)
$Z_A a$
$Z_A b$
$M_t^{\oplus}$
Z_A
$Z_A b$
$Z_A a$
$M_t^{\oplus}$
(b)
Θ_A
$M_t^{\oplus}$
Θ_A
Θ_A
(c)
Φ_A
$M_t^{\oplus}$
Φ_A
$M_t^{\ominus}$
Φ_A
(d)
I
Ia
(e)

Figure 3: XXX

where

$$J = \frac{\pi d^3 t}{8}, \qquad K_\mathrm{t} = \frac{\pi d^3 t}{4}, \qquad G = \frac{E}{2\,(1+\nu)};$$

each beam branch contribution is finally accumulated to obtain the overall structure strain energy, possibly split into its symmetric and skew-symmetric parts.

Once the structure strain energy has been evaluated, we cast a system of equations through which we impose a null deflection in A, namely

$$\frac{\partial U}{\partial X_\mathrm{A}} = 0 \qquad \frac{\partial U}{\partial Y_\mathrm{A}} = 0 \qquad \frac{\partial U}{\partial \Psi_\mathrm{A}} = 0 \tag{2}$$

$$\frac{\partial U}{\partial Z_\mathrm{A}} = 0 \qquad \frac{\partial U}{\partial \Theta_\mathrm{A}} = 0 \qquad \frac{\partial U}{\partial \Phi_\mathrm{A}} = 0. \tag{3}$$

The value of the six unknown reactions at A may be then derived as a (linear) function of the remaining loads, e.g.

$$\begin{aligned} X_\mathrm{A} &= X_\mathrm{A}\,(F,P,H,I) = \alpha F + \beta P + \gamma H + \delta I \\ Y_\mathrm{A} &= Y_\mathrm{A}\,(F,P,H,I) = \ldots \\ \Psi_\mathrm{A} &= \ldots \end{aligned}$$

etc., where the linear combination coefficients are placeholders for their actual counterpart, which derive from the system solution.

Once obtained expressions for the constraint reactions in A, we substitute them within the structure strain energy expression, thus deriving for the latter an form which depends on the external loads alone, i.e.

$$\begin{aligned} U &= U\,(P,F,X_\mathrm{A},Y_\mathrm{A},\Psi_\mathrm{A},H,I,Z_\mathrm{A},\Theta_\mathrm{A},\Phi_\mathrm{A}) \\ &= U\,(P,F,X_\mathrm{A}(P,F,H,I),\ldots,\Phi_\mathrm{A}(P,F,H,I)) \\ &= U\,(P,F,H,I)\,. \end{aligned}$$

All the contribution of the external loads to the structure strain energy are now made explicit - they could formally remain nested within the constraint reaction symbols, but at the risk of leaving them behind while performing the differentiation[5], and we may proceed in evaluating the requested deflections as

$$\tilde{u}_{\mathrm{B}} = \left.\frac{\partial U}{\partial F}\right|_{F=0,I=0} \qquad \tilde{u}_{\mathrm{C}} = \left.\frac{\partial U}{\partial P}\right|_{F=0,I=0}$$
$$\tilde{w}_{\mathrm{C}} = \left.\frac{\partial U}{\partial H}\right|_{F=0,I=0} \qquad \tilde{w}_{\mathrm{D}} = \left.\frac{\partial U}{\partial I}\right|_{F=0,I=0},$$

where the fictitious nature of the F, I loads is finally declared.

The constraint reaction components at A may be derived by substituting the actual null value of F, I in their previously obtained expressions; their counterpart at E may be derived by casting and solving the equilibrium equations for the whole principal structure, now that all the therein applied loads are known.

[5]it actually happens with the Maxima algebraic manipulator if the constraint reaction compontents are not *explicitly* declared dependent on them