Advanced non-linear analysis

Code_Aster, Salome-Meca course material

EDF

Outline

- Description of non-linear problems
- Theoretical elements for solving non-linear problems
- Solving non-linear problems with Newton
- Using *Code_Aster*

What is a non-linear problem in mechanics ?

Non-linear problems

- Non-linearities come from :
  - Kinematic (movement and strains) : large displacement, large rotation, large strains
  - Material : non-linear response, history-dependent response
  - Contact/friction
- The three non-linearities should been coupled
- Numerical simulation require expert analysis

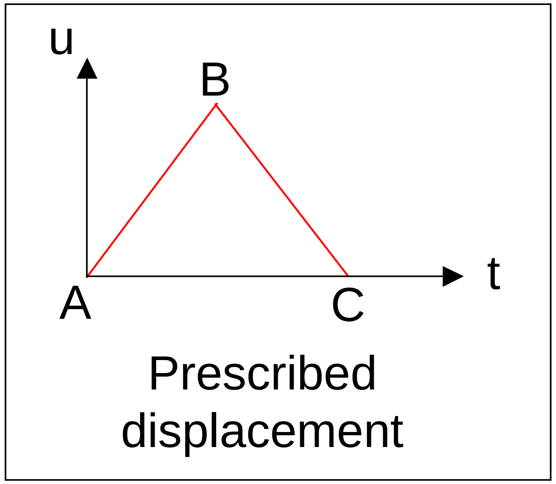
Non-linear material

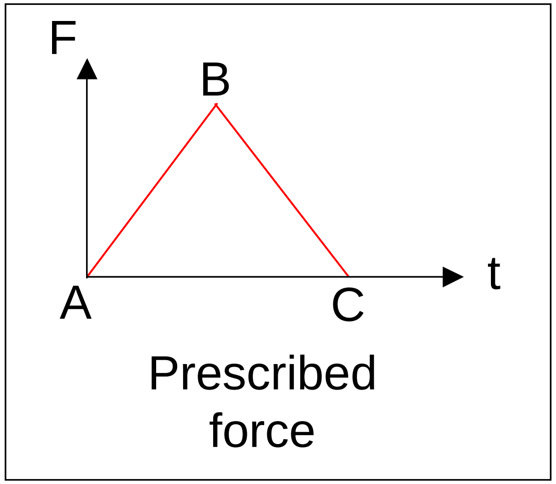
- Material non-linearity
  - Experimental identification : identify macroscopical response force/displacement

  - Most materials are non-linear and/or history dependent

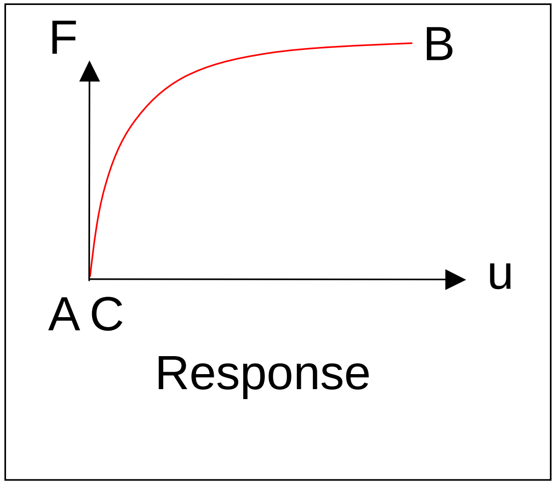
Non-linear material

- Displacement/force function is not linear ***but not*** history dependent





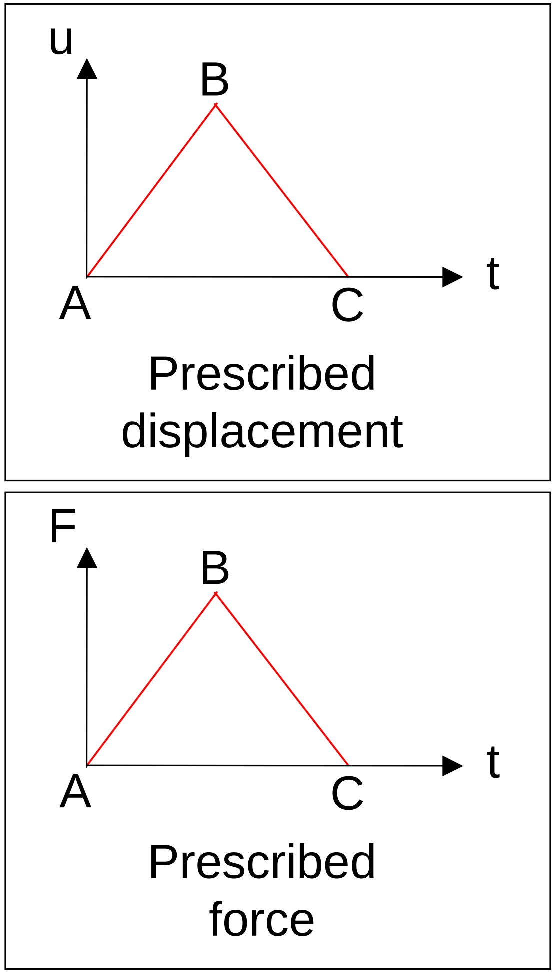
$$F = \varphi(u) \quad \text{with} \quad \varphi(u) \neq a.u + b$$



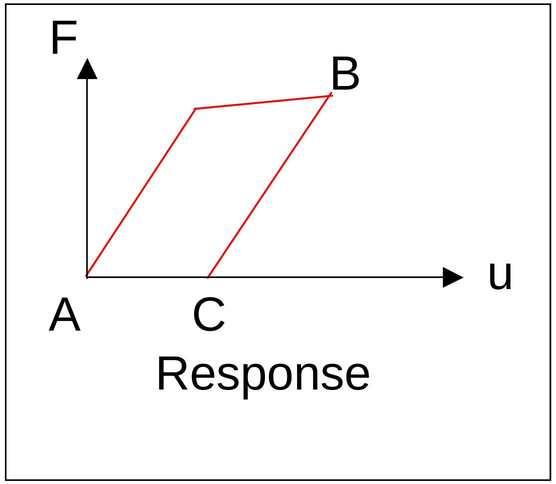
$$u(C) = u(A)$$
$$F(C) = F(A)$$

Non-linear material

- Displacement/force function is not linear ***and*** history dependent



$$F = \varphi(u) \quad \text{with} \quad \varphi(u) \neq a.u + b$$

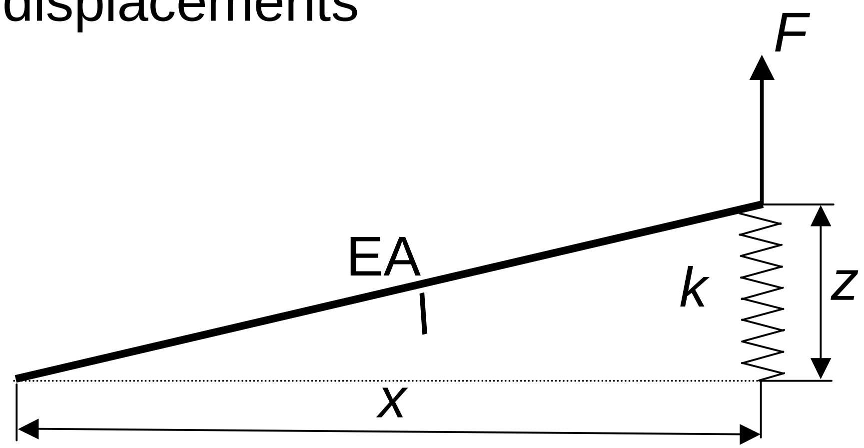


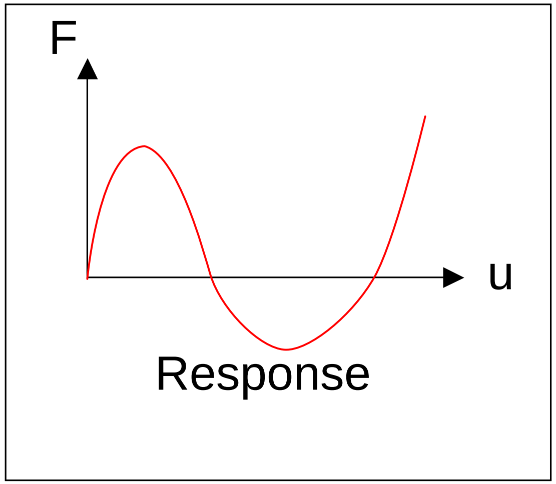
$$u(C) \neq u(A)$$

Residual displacement
Next loading : initial displacement $\neq 0$

Non-linear kinematic

- Truss/beams/shells elements with large rotation and/or large displacements

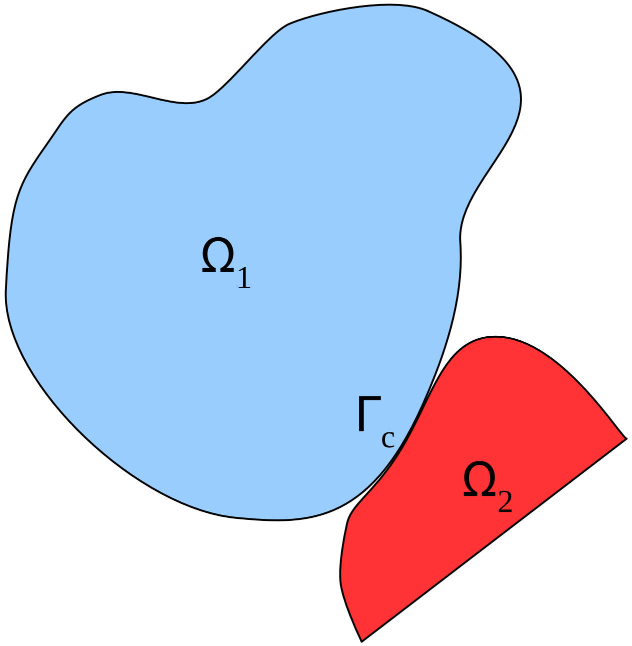


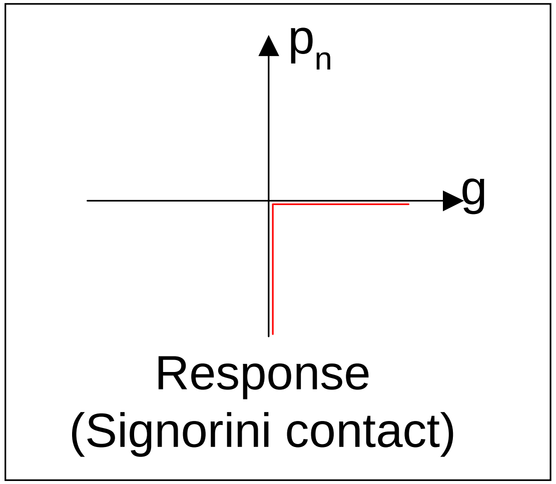


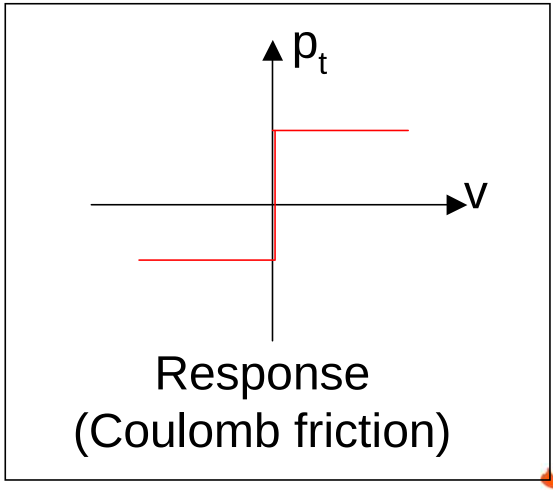
E: Young modulus
A: section
L: length
F: load
u: displacement
k: rigidity of the spring

Non-linear contact/friction

- Contact and friction: a very difficult non-linear problem





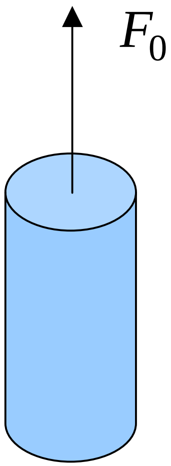


p_n: normal pressure
p_t: tangent pressure
g: distance between solids (gap)
v: tangential speed between solids

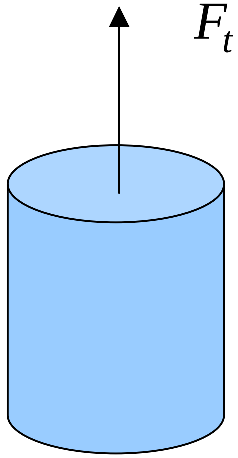
Some theoretical elements

Equations – Continuous form

- Measure of stress : Cauchy (true) stress $$\sigma = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S}$$



$$\sigma_0 = \lim_{\Delta S_0 \to 0} \frac{\Delta F_0}{\Delta S_0}$$



$$\sigma_t = \lim_{\Delta S_t \to 0} \frac{\Delta F_t}{\Delta S_t}$$

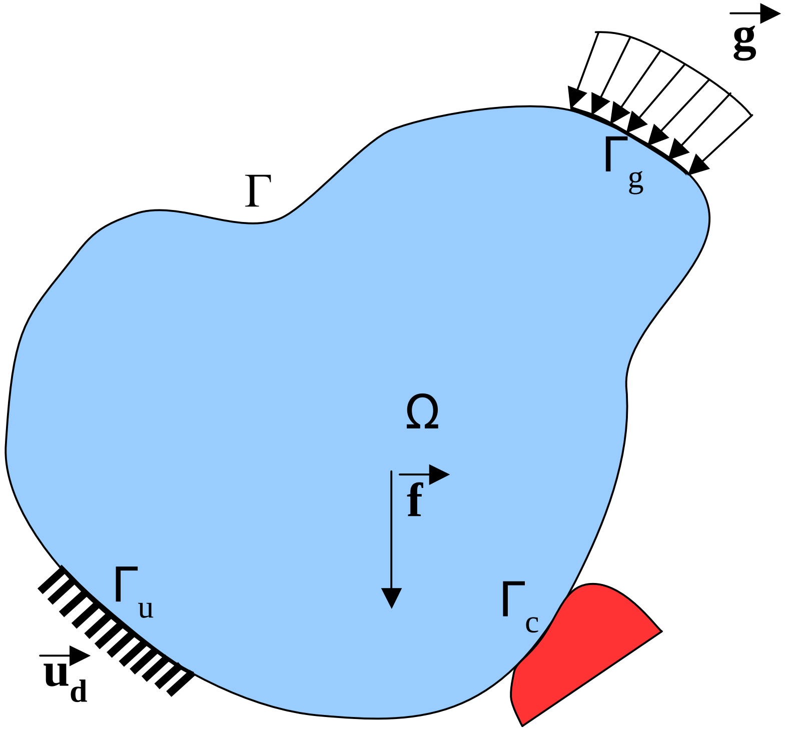
Cauchy stress : measure on ***deformed*** configuration

- Measure of strain :
  - for finite strain, it's an arbirtray choice, depending on behavior law
  - For small strain $$\varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla^t u\right)$$

Equations – Continuous form

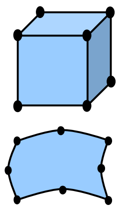
- Defining solid Ω which is in equilibrium with external forces
  - Γ is external boundary
  - Γ_u to apply prescribed displacement $\mathbf{u_d}$
  - Γ_g to apply prescribed force $\mathbf{g}$
  - Γ_c to define contact/friction
  - **f** is volumic force

$$\begin{cases} div\sigma + f = 0 & in \quad \Omega \\ \sigma.n = g & on \quad \Gamma_g \\ u = u_d & on \quad \Gamma_u \\ L_{cf}(u) & on \quad \Gamma_c \end{cases}$$



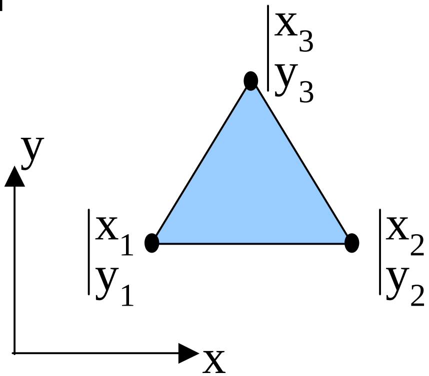
Equations – Finite element approximation

- Main shapes for elements :



  - segment, triangle, quadrangle, hexaedron, tetraedron, pyramid, prism
  - linear or quadratic (curved or straight edges)

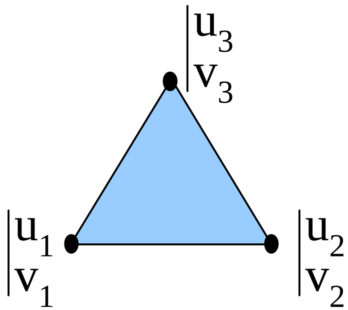
- Geometry discretization :
  - Coordinates in element $\{x\}$
  - Discretized coordinates at nodes $\{x_i\}$
  - Shape functions $\lfloor N_g \rfloor$



$$\{x\} = \lfloor N_g \rfloor \{x_i\}$$

Equations – Finite element approximation

- Discretization of the unknowns:
  - Unknowns depending on physic : displacement for mechanics temperature for thermic, ...
  - Displacements in element $\{u\}$
  - Discretized displacement at nodes $\{u_i\}$
  - Shape functions $[N_u]$



$$\{u\} = [N_u].\{u_i\}$$

We suppose isoparametric elements $[N] = [N_g] = [N_u]$

Equations – Finite element approximation

- Weak form of equilibrium (weighted residuals) :
  - Find $u \in E_u$ with $\forall \tilde{u} \in E_{\tilde{u}}$:

$$\int_{\Omega} \sigma(u) : \varepsilon(\tilde{u}).d\Omega = \int_{\Omega} f(u) : \tilde{u}.d\Omega + \int_{\Gamma_g} g(u) : \tilde{u}.d\Gamma$$

  - Virtual displacements $\tilde{u}$ and virtual strains $\varepsilon(\tilde{u}) = \frac{1}{2}\left(\nabla\tilde{u} + \nabla^t\tilde{u}\right)$
  - With E_u space of kinematically admissible displacements :

$$E_u = \{u \quad with \quad u(X) = u_d \quad \forall X \in \Gamma_u\}$$

  - With $E_{\tilde{u}}$ space of virtual displacements :

$$E_{\tilde{u}} = \{\tilde{u} \quad with \quad \tilde{u}(X) = 0 \quad \forall X \in \Gamma_u\}$$

- The weak form is the mechanical concept of ***virtual power***
- Galerkin form : $\{u\} = [N].\{u_i\}$ and $\{\tilde{u}\} = [N].\{\tilde{u}_i\}$

Equations – Finite element approximation

- Non-linear problem : stress depending on displacement $\sigma(u)$
  - From displacement ***u*** to strain ε : **kinematic** non-linearity
  - From strain ε to stress σ : **material** non-linearity

- Non-linear material

$$\sigma(u) = \sigma(\varepsilon(u))$$

- Non-linear material depending on material's history (plasticity,...) :

$$\sigma(u) = \sigma(\alpha, \varepsilon(u))$$

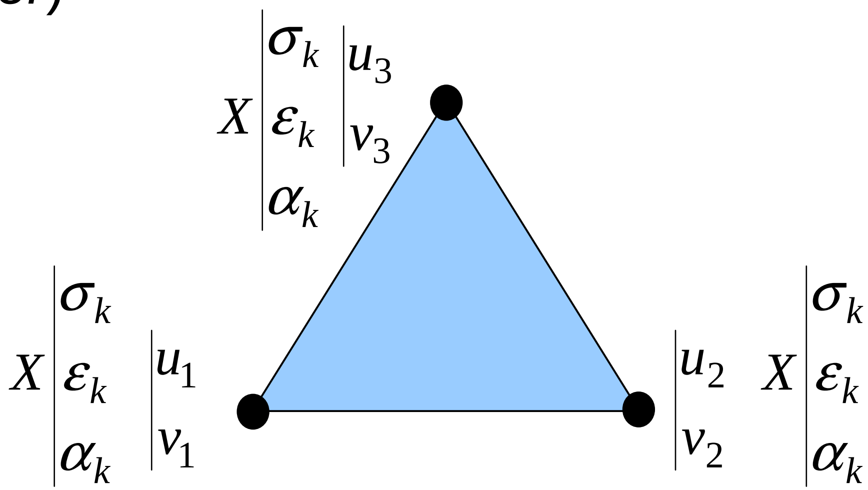
Internal variables α

Equations – Finite element approximation

- Numerical quadrature : to compute continuous som, using a Gauss approximation

$$\int_{\Omega} \sigma . d\Omega \rightarrow \sum_{k=1}^{\text{nb points}} \sigma_{\mathrm{k}} . \omega_k$$

- Stress, strain and internal variables : discretized at ***numerical points*** X (quantities post_fixed by _ELGA in *Code_Aster*)
- Displacement : discretized at ***nodes*** ● (quantities post_fixed by **_NOEU** in *Code_Aster*)



Equations – Finite element approximation

- Computation of internal forces

$$\int_{\Omega} \sigma(u) : \varepsilon(\tilde{u}).d\Omega \rightarrow \{L_{\text{int}}(u)\} = [Q(u)].\{\sigma(u)\}$$

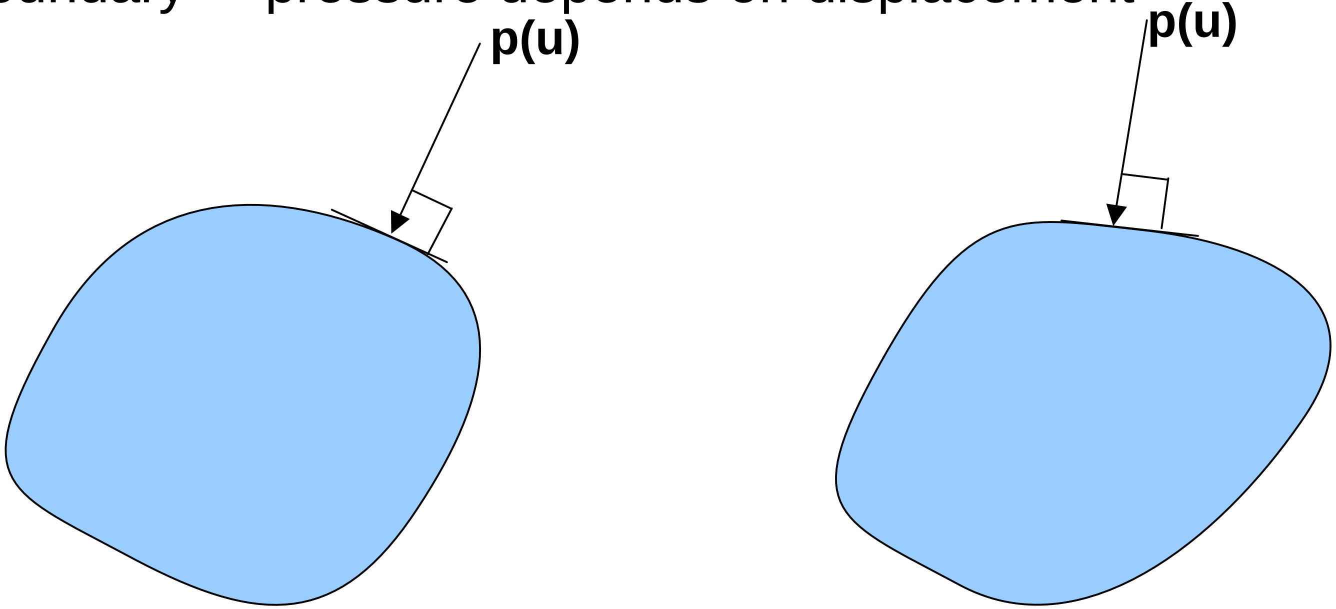
- Non-linear internal forces : depending on u
  - Non-linear kinematic $[Q(u)]$
  - Non-linear material $\{\sigma(u)\}$

Equations – Finite element approximation

- Computation of external forces

$$\int_{\Omega} f(u):\tilde{u}.d\Omega + \int_{\Gamma_g} g(u):\tilde{u}.d\Gamma \rightarrow \{L_{ext}(u)\}$$

- Non-linear external forces : depending on u
  - Non-linear when following forces : pressure always normal to boundary → pressure depends on displacement



Equations – Finite element approximation

- Final equilibrium equation : discretized form

$$\{L_{\text{int}}(u)\} - \{L_{ext}(u)\} = \{0\}$$

Non-linear equation, depending on u

General algorithm for non-linear problem

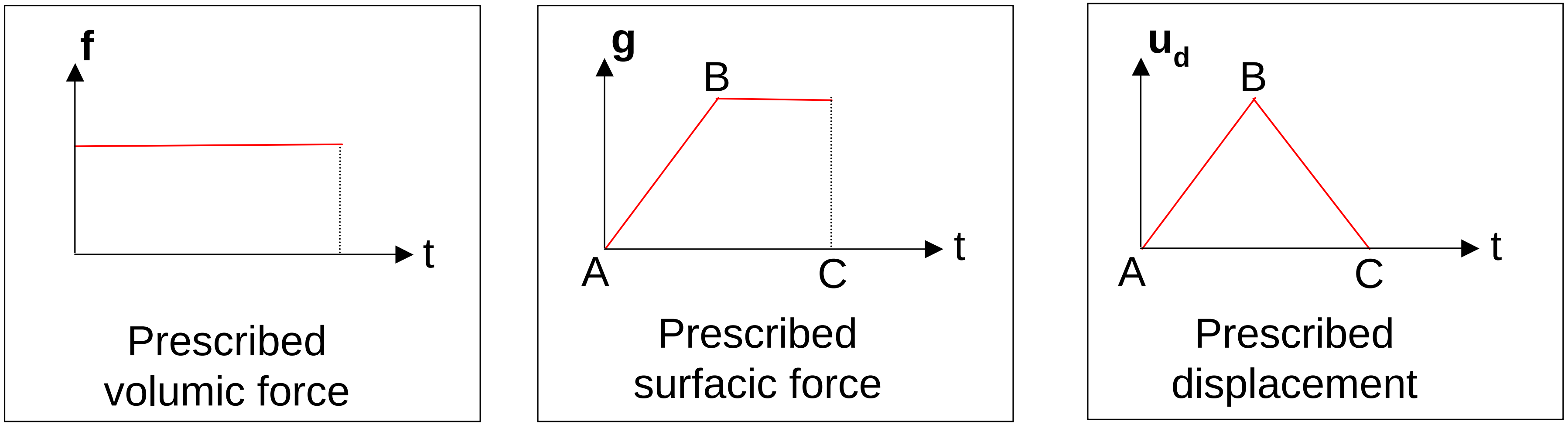
Solving non-linear problem

- To solve non-linear problem : an incremental algorithm
  - Problem is parametrized by the parameter *t*
  - *t* **is not real time** (quasi-static problem)

- Why parametrization ?
  - Real boundary conditions should been applied by non-constant values
  - Non-linear material produced non-linear equation, precision should depends on incremental step size
  - An non-linear problem is easier to solve when cut by small step size

Solving non-linear problem

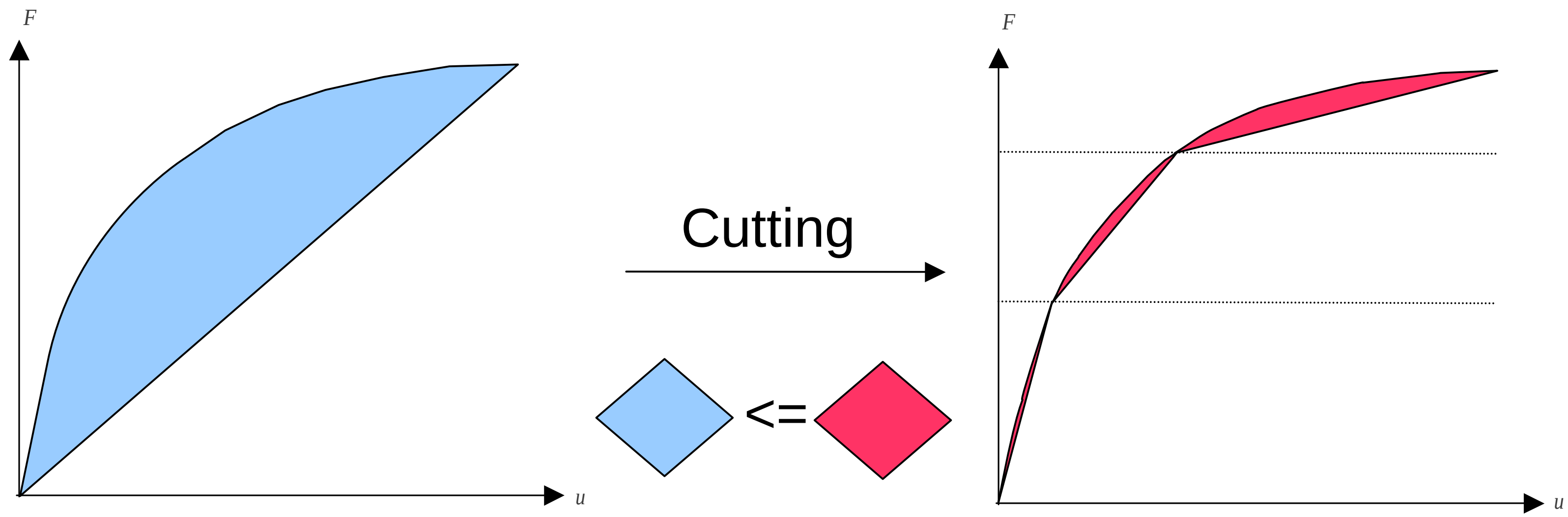
- Parametrization because boundary conditions should been applied by non-constant values



Some examples of prescribed boundary conditions

Solving non-linear problem

- Parametrization to cut a non-linear problem to decrease degree of non-linearity



High degree of non-linearity on complete problem

Low degree of non-linearity at each step

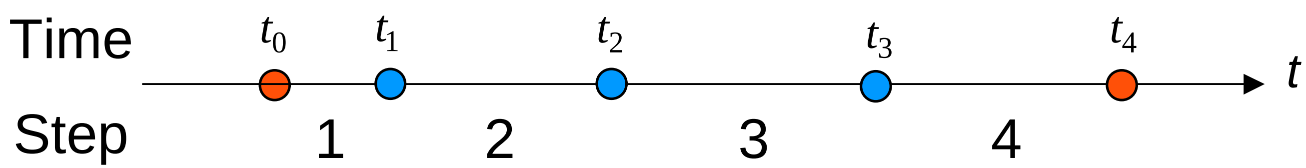
Solving non-linear problem

- The parameter *t* in the equations :
  - Internal forces : dependence with respect to *t* is **implicit**, it results from the integration of the constitutive relation in time
  - External forces : dependence with respect to *t* is **explicit** (applied BC) or **implicit** (following forces)

$$\{L_{\mathrm{int}}(u(t))\} - \{L_{ext}(t, u(t))\} = \{0\}$$

- « Time » discretization, all quantities are parametrized by step *k* :

$$L_k^{\mathrm{int}} \quad L_k^{\mathrm{ext}} \quad u_k \quad t_k \quad \sigma_k \quad \varepsilon_k \quad \alpha_k$$



Incremental quantities : $u_k = u_{k-1} + \Delta u$

Solving non-linear problem

- Solving a non-linear equation $F(x)=0$: the Newton's method
  - Construction of the series $\left(x^{n}\right)_{n\in N}$
  - Taylor series expansion of the first order

$$0=F(x^{n})\approx F(x^{n-1})+F'(x^{n-1}).(x^{n}-x^{n-1})$$

  - Next value of the term of the series

$$x^{n}=x^{n-1}-\frac{1}{F'(x^{n-1})}F(x^{n-1})$$

- Properties :
  - Quadratic convergence near the solution
  - Computation of $F'(x^{n-1})$ should be very expensive
  - $F'(x^{n-1})\neq 0$

Solving non-linear problem

- Newton's method for equilibrium equation, at step *k* :

$$\{L_{\text{int}}(u_k(t_k))\} - \{L_{ext}(t_k, u_k(t_k))\} = \{0\}$$

$$\{L_{\text{int}}(u(t))\} - \{L_{ext}(t, u(t))\} = \{0\}$$

Simplify : we are at step *k*

- Internal forces :

$$L^{\text{int},n} \approx L^{\text{int},n-1} + \left(\frac{\partial L^{\text{int}}}{\partial u}\right)_{u^{n-1}} .\delta u^n \quad \text{with} \quad \delta u^n = (u^n - u^{n-1})$$

- External forces :

$$L^{ext,n} \approx L^{ext,n-1} + \left(\frac{\partial L^{ext}}{\partial u}\right)_{u^{n-1}} .\delta u^n \quad \text{with} \quad \delta u^n = (u^n - u^{n-1})$$

Solving non-linear problem

- Newton's method for equilibrium equation, at step *k* :

$$L^{\text{int},n-1} - L^{ext,n-1} + \left(\frac{\partial L^{ext}}{\partial u} - \frac{\partial L^{\text{int}}}{\partial u} \right)_{u^{n-1}} .\delta u^{n} = 0$$

$$K^{n-1}.\delta u^{n} = R^{n-1}$$

- K^{n-1} is tangent matrix

$$K^{n-1} = \left(\frac{\partial L^{\text{int}}}{\partial u} \right)_{u^{n-1}} - \left(\frac{\partial L^{ext}}{\partial u} \right)_{u^{n-1}}$$

- R^{n-1} is equilibrium residual

$$R^{n-1} = \left(L^{ext,n-1} - L^{\text{int},n-1} \right)$$

Solving non-linear problem

- Global algorithm :
  1) Compute internal and external forces
  2) Compute tangent matrix
  3) Solve linear system
  4) Update displacements
  5) Evaluate convergence

Solving non-linear problem

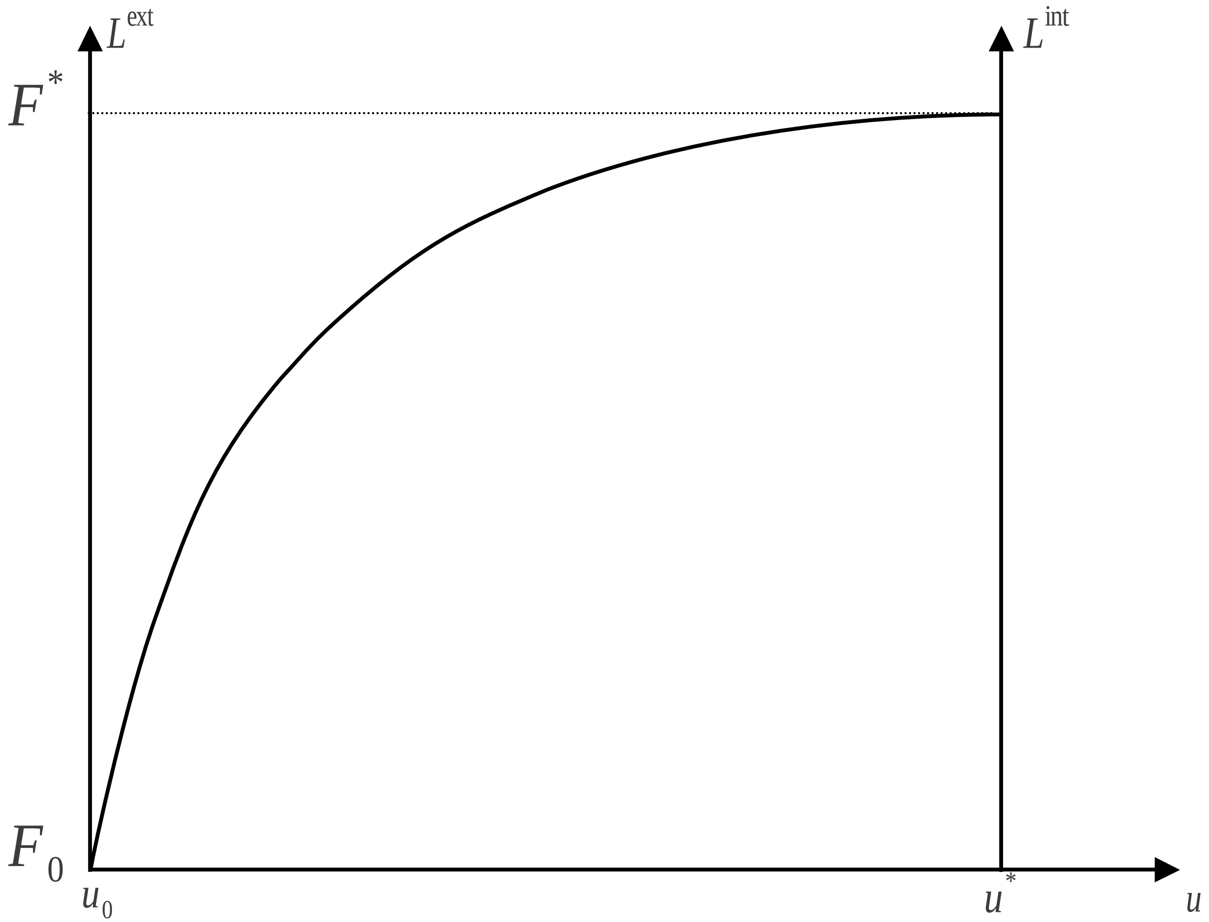
- Compute internal and external forces :
  - Exact computation is **necessary** (software requirement)
  - Non-linear behavior is evaluated at each Gauss point
  - Complex behavior laws computation should be expensive
  - Precision of computation of some behavior laws should depend on increment of displacement :
    - Visco-plasticity
    - Non-radial loading
    - `PETIT_REAC` strain measure

Solving non-linear problem

- Compute tangent matrix :
  - Exact computation is NOT necessary
  - Tangent matrix is evaluated at each Gauss point
  - Tangent matrix computation should be expensive

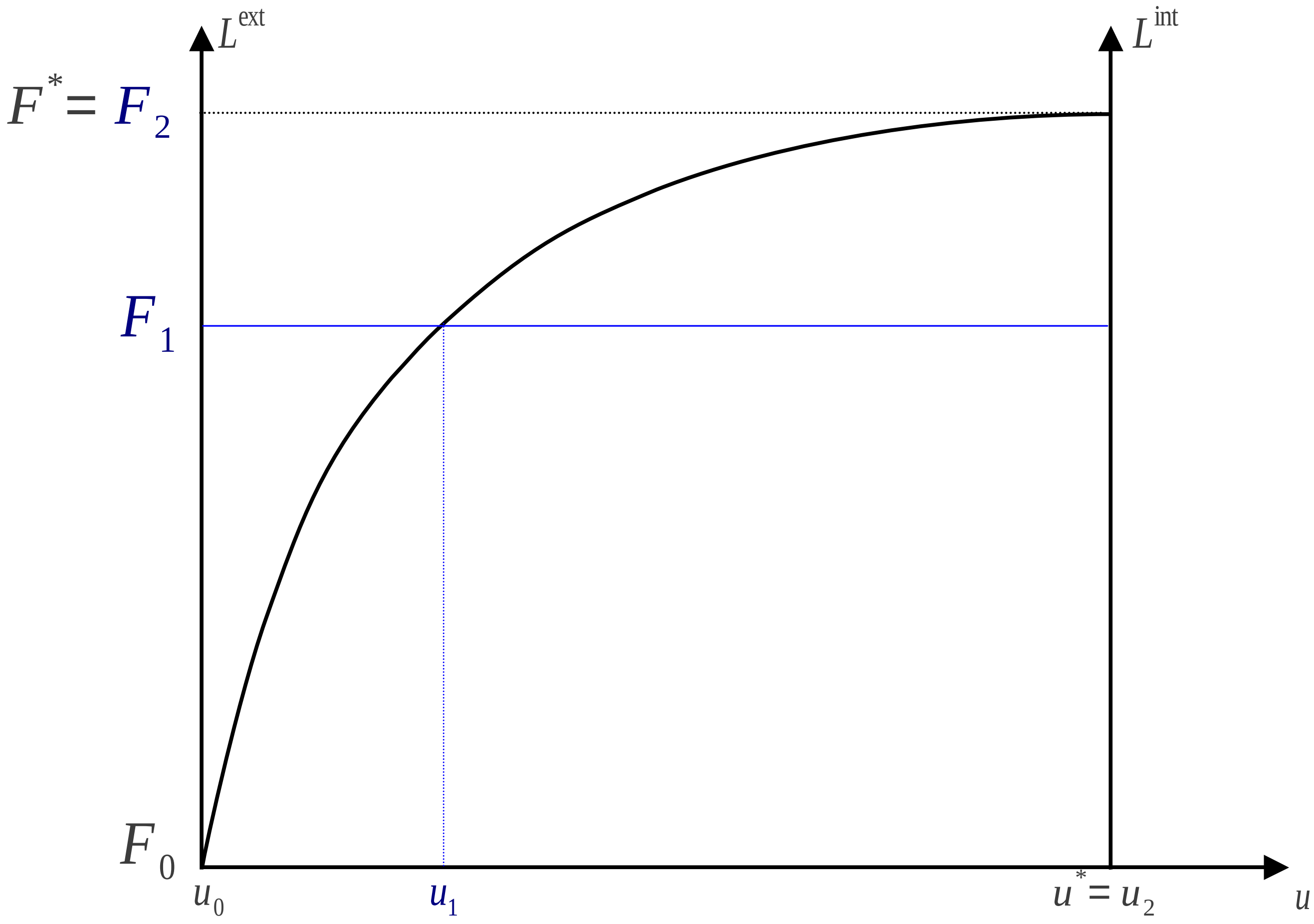
Solving non-linear problem – Newton by graphic

- Find the solution (u^*, F^*) where F^* is known and u^* is unknown



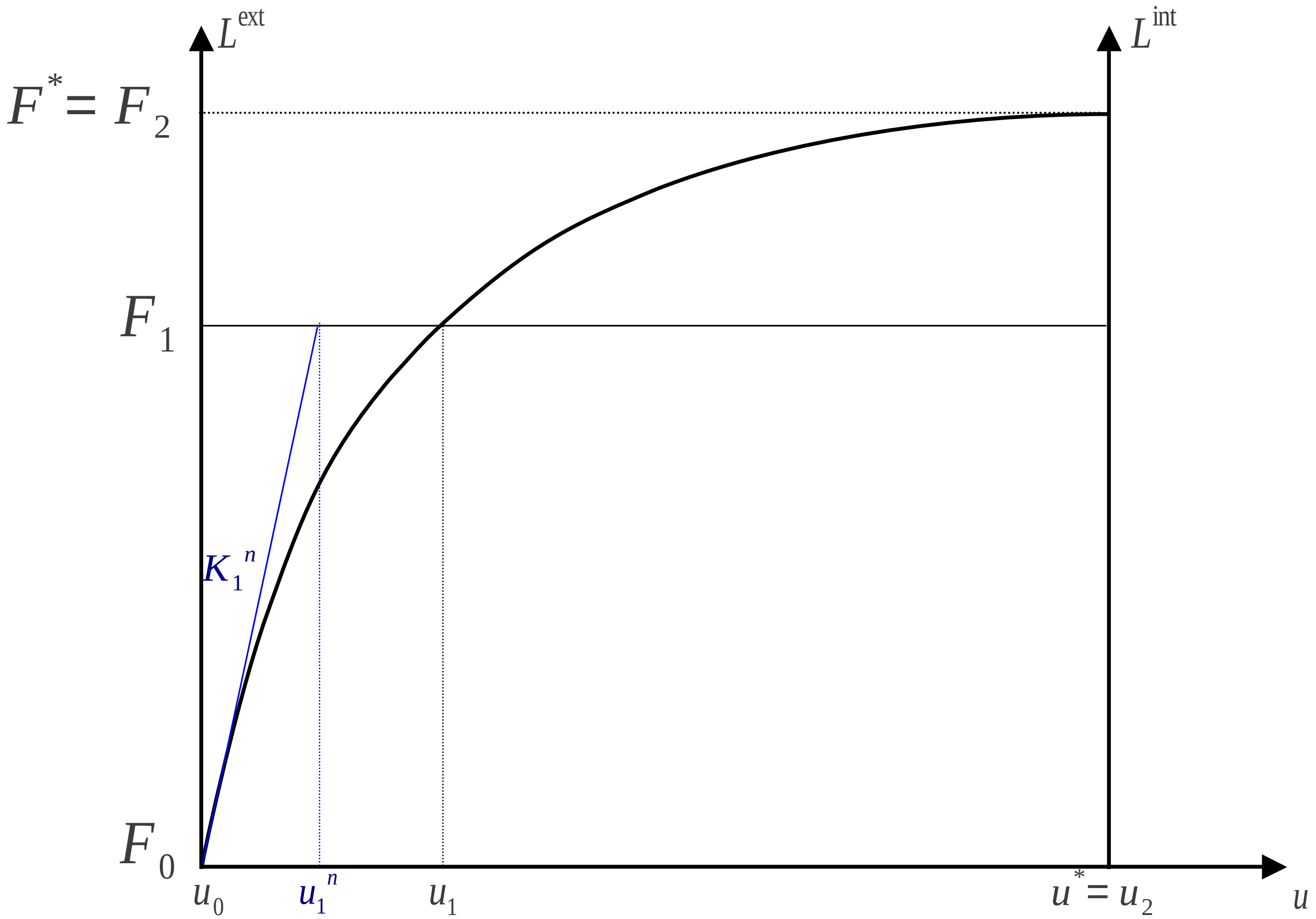
Solving non-linear problem – Newton by graphic

- Dividing F^* in two increments (reducing degree of non-linearity) : find (u_1, F_1) and (u_2, F_2)



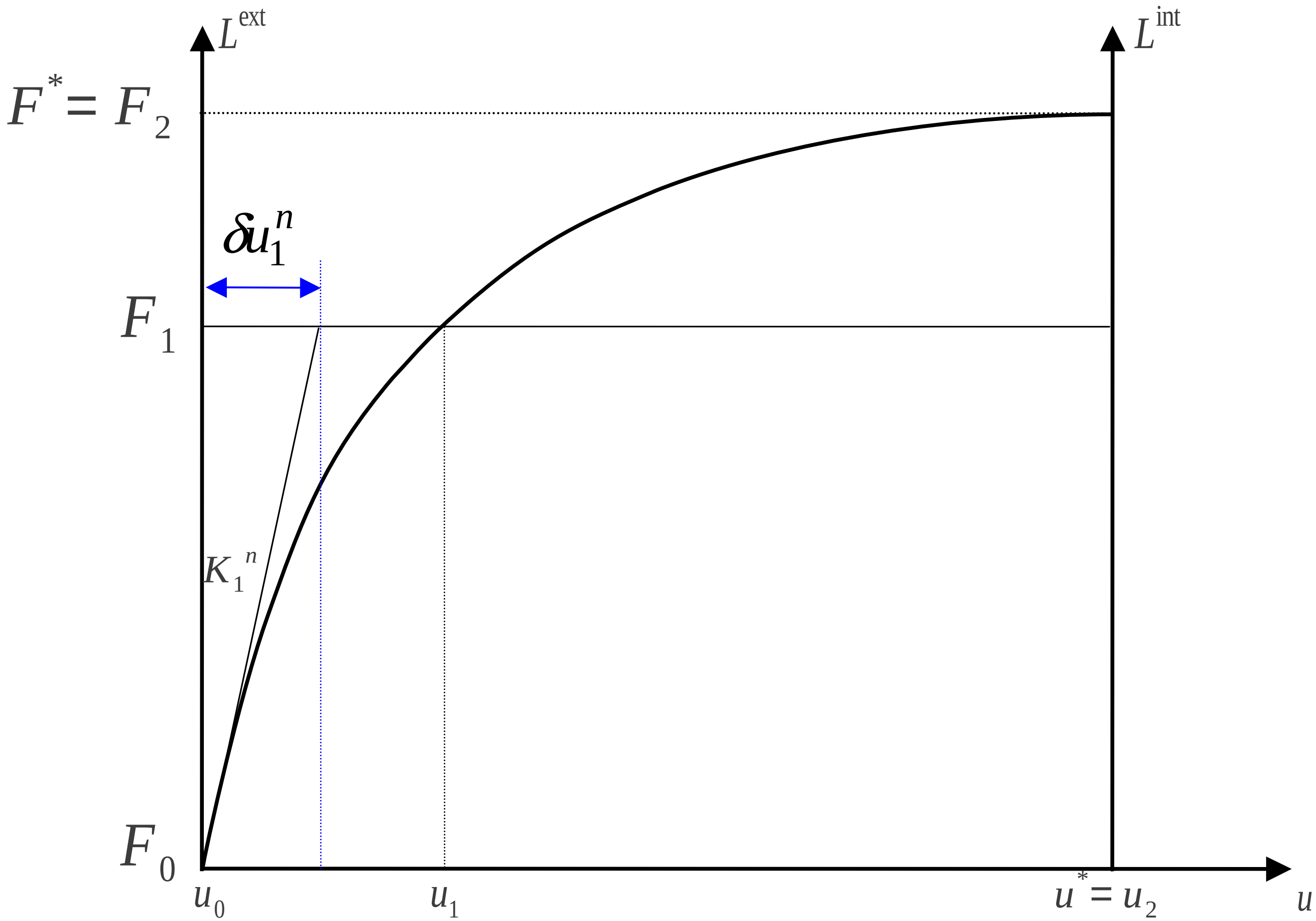
Solving non-linear problem – Newton by graphic

- Compute tangent matrix K_1^n for iteration Newton *n* and step *1*



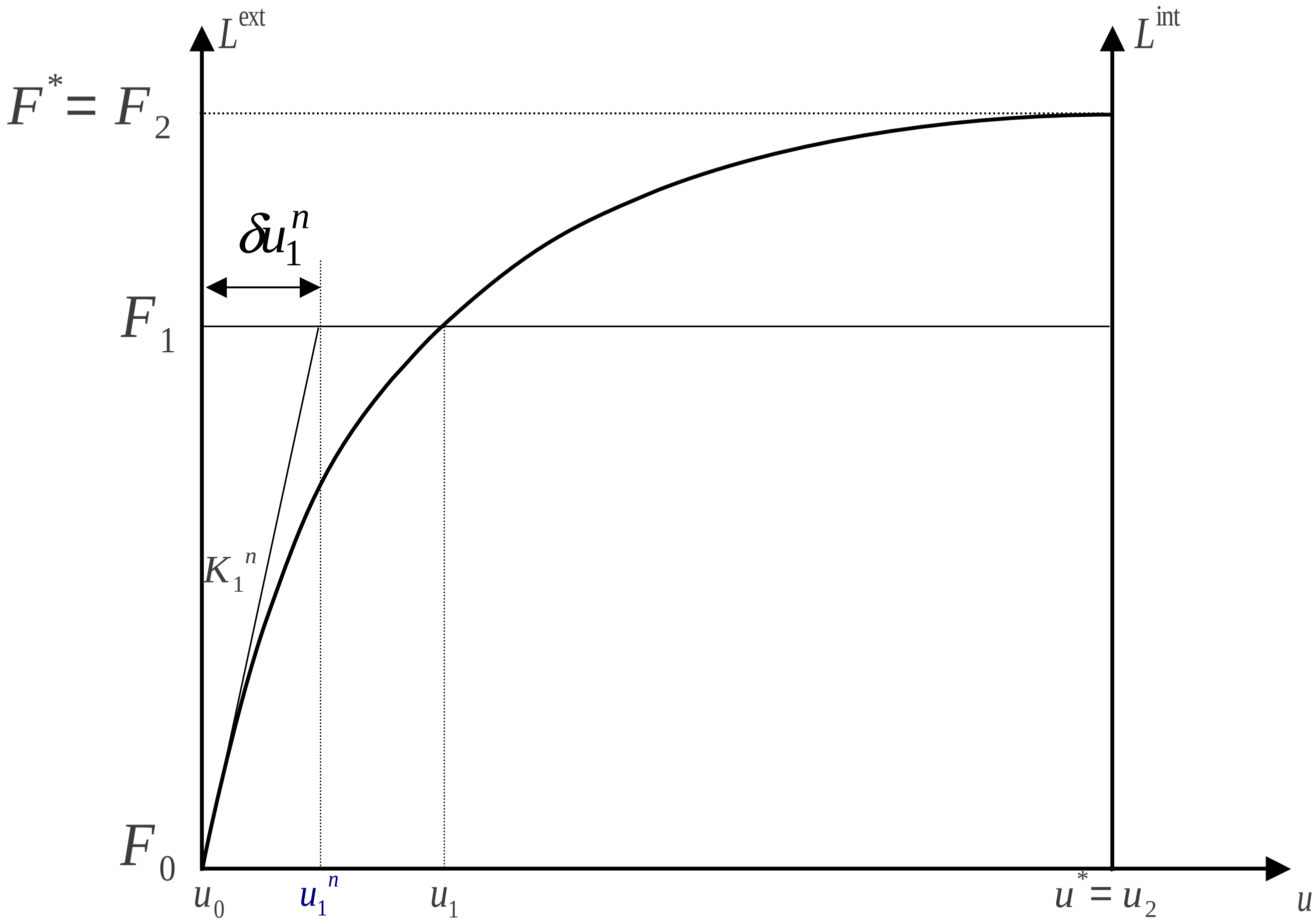
Solving non-linear problem – Newton by graphic

- Solving $K_1^n.\delta u_1^n = F_1 - F_0 \rightarrow \delta u_1^n$



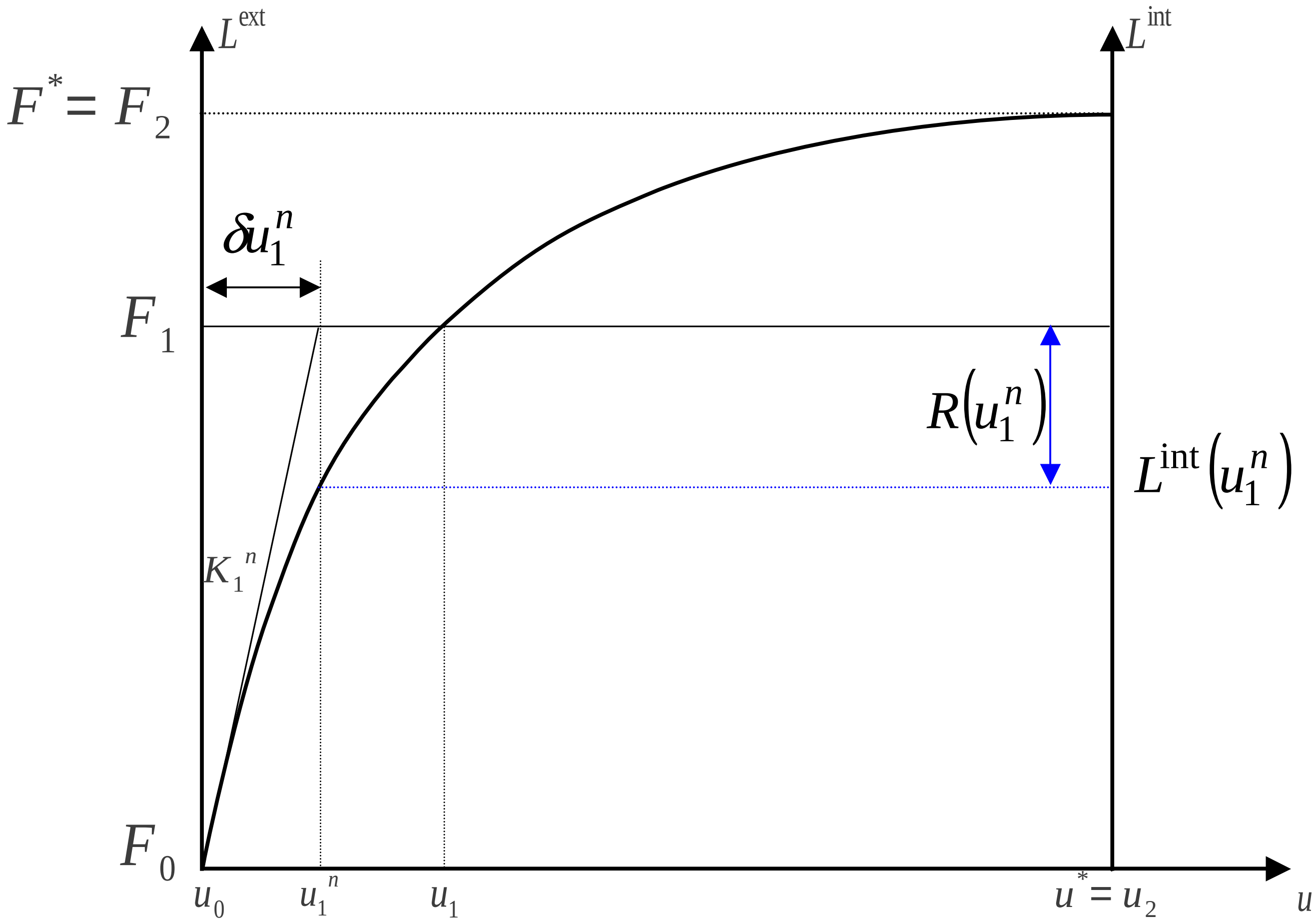
Solving non-linear problem – Newton by graphic

- Update displacements $u_1^n = u_1^{n-1} + \delta u_1^n$

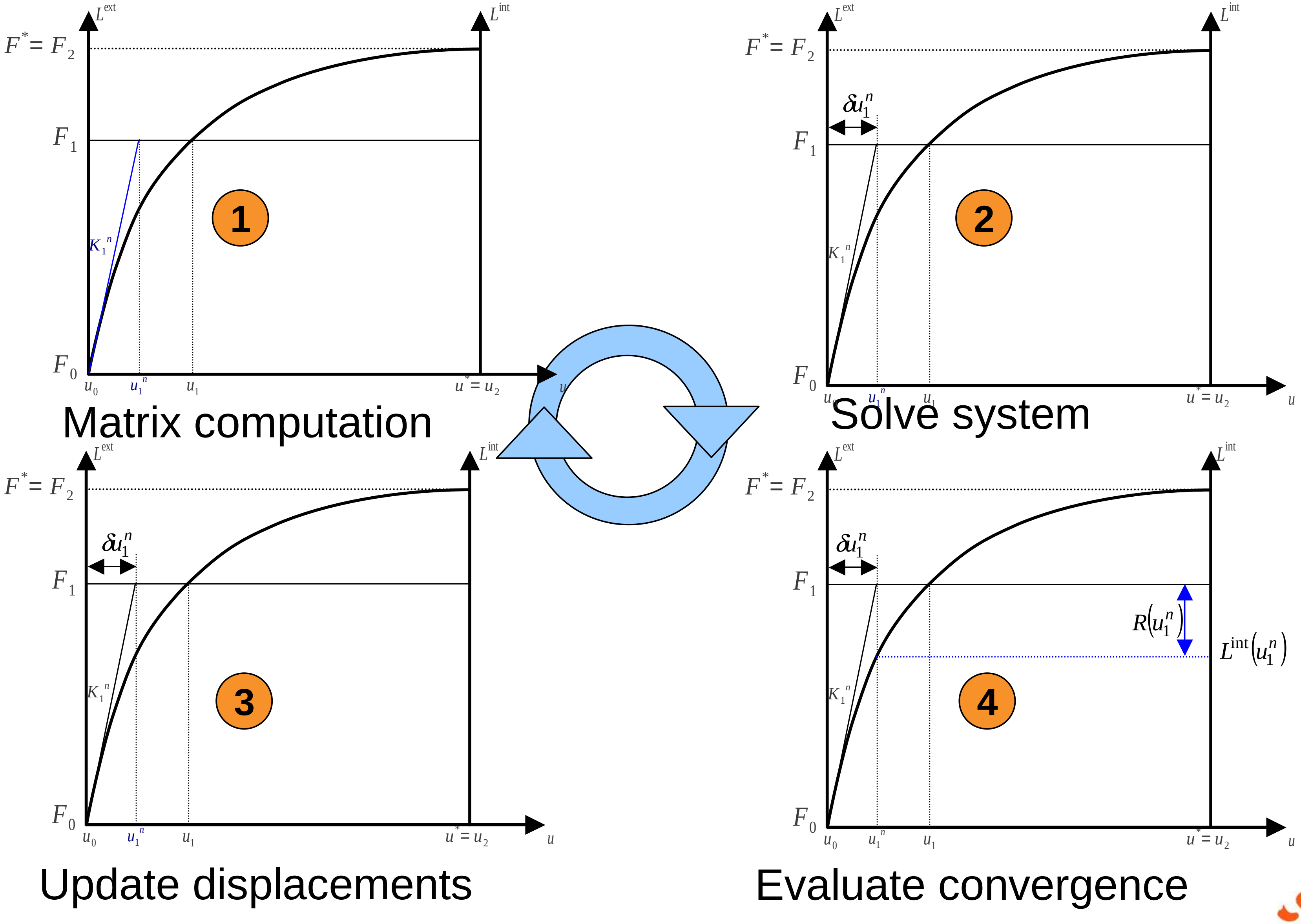


Solving non-linear problem – Newton by graphic

- Compute $L^{\text{int}}\left(u_1^n\right)$ and $R = F_1 - L^{\text{int}}\left(u_1^n\right)$



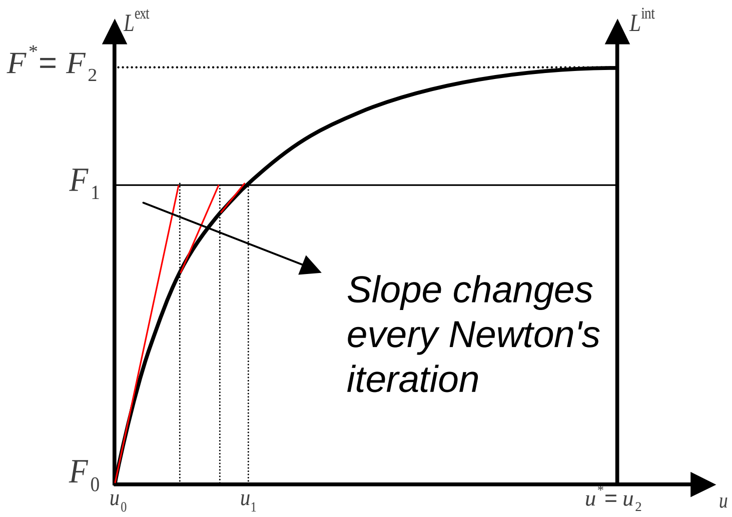
Solving non-linear problem – Newton by graphic



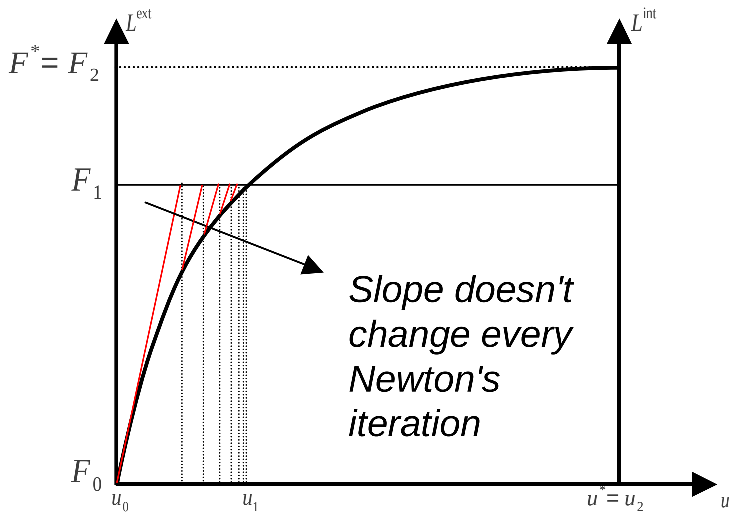
Solving non-linear problem

1 Matrix computation :

- True Newton method : K_i^n is evaluate every step and every Newton iteration
- Quasi-Newton method : K_i^n is evaluate every *i* Newton's iteration and every *n* time's step



True Newton method

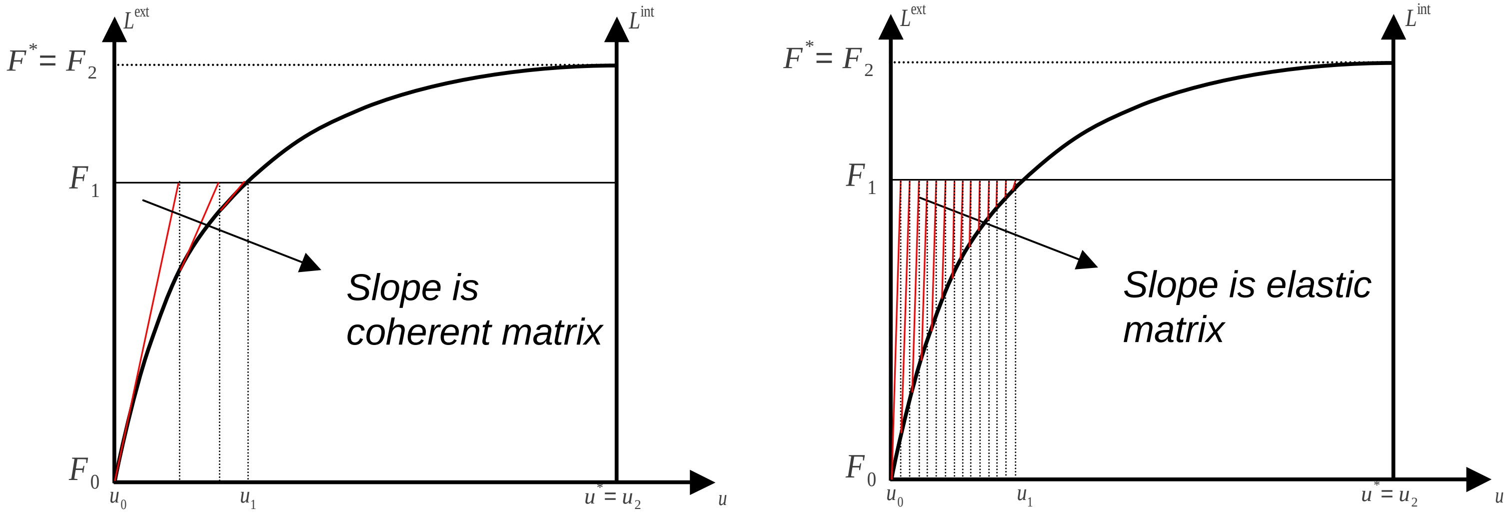


Quasi Newton method

Solving non-linear problem

1 Matrix computation :

- Quasi-Newton method : compute approximate matrix (elastic matrix)



True Newton method

Quasi Newton method

Solving non-linear problem

1 Matrix computation : Why quasi-Newton methods ?

- Compute exact matrix every iteration : matrix must been factorized → very expensive
- Make more iterations but each iteration is quicker
- Elastic matrix achieve convergence for all standard generalized materials with a lot of iterations (thousands). Elastic matrix is computed and factorized ONE time : very cheap
- Elastic matrix is the better choice for unloading loading's cases

Solving non-linear problem

2 Solve system : using direct solver or iterative solver

Solving non-linear problem

3 Update displacement - Improving by line-search - **Method**

- Solving $K^{n-1}.\delta u^{n} = R^{n-1}$ is equivalent to ***minimizing*** functional
- Solving $K^{n-1}.\delta u^{n} = R^{n-1}$ give the ***direction*** δu^{n} of the solution
- Solving functionnal minimisation give the ***line-search coefficient*** ρ
- Update displacements

$$u^{n} = u^{n-1} + \rho.\delta u^{n}$$

Solving non-linear problem

3 Update displacement - Improving by line-search - **Functional**

- Functional (scalar) *J* is :

$$u \rightarrow J(u) = \int_{\Omega} \Phi(\varepsilon(u)).d\Omega - \int_{\Omega} f.u.d\Omega - \int_{\Gamma} t.u.d\Gamma$$

- With $\Phi(\varepsilon(u))$ is the density of free energy, for hyperelastic material :

$$\varepsilon = \frac{\partial \Phi}{\partial \sigma}$$

- This functional is ***convex,*** minimizing convex function → its gradient ***vanishes*** :

$$\nabla J(u).\tilde{u} = 0 \quad \forall \tilde{u} \in E_{\tilde{u}}$$

- Gradient vanishes <=> Virtual power

$$\int_{\Omega} \sigma(u) : \varepsilon(\tilde{u}).d\Omega = \int_{\Omega} f(u) : \tilde{u}.d\Omega + \int_{\Gamma_g} g(u) : \tilde{u}.d\Gamma \quad \forall \tilde{u} \in E_{\tilde{u}}$$

- Find equilibrium : ***minimize** J* ***scalar functional***

Solving non-linear problem

3 Update displacement - Improving by line-search - **Minimize functional**

- Derivative of the functional

$$\left\{L_{\text{int}}(u^{n-1}+\rho\delta u^{n})\right\}-\left\{L_{ext}(u^{n-1}+\rho\delta u^{n})\right\}=\{0\}$$

- Scalar functional

$$g(\rho)=\left\langle \delta u^{n}\right\rangle.\left\{L_{\text{int}}(u^{n-1}+\rho\delta u^{n})\right\}-\left\{L_{ext}(u^{n-1}+\rho\delta u^{n})\right\}=\{0\}$$

- Using classical method for minimizing scalar convex function (dichotomy for instance)

Solving non-linear problem

3 Update displacement - Improving by line-search - **Notes**

- Only few iterations : it's not necessary to find *exact* zero of g
- Only compute internal and external forces (no matrix)
- Algorithms for minimizing scalar function are very efficient and very simple

Solving non-linear problem

4 Evaluate convergence

- Absolute (**RESI_GLOB_MAXI**)

$$\left\| L^{ext} - L^{\mathrm{int},n-1} \right\|_{\infty} \leq \zeta$$

- Relative (**RESI_GLOB_RELA**)

$$\frac{\left\| L^{ext} - L^{\mathrm{int},n-1} \right\|_{\infty}}{\left\| L^{ext} \right\|_{\infty}} \leq \zeta$$

- By reference : giving stress reference (**RESI_REFE_RELA**)

$$\left\| L^{ext} - L^{\mathrm{int},n-1} \right\|_{k} \leq \gamma . L_k^{ref}$$

Using non-linear in Code_Aster

Non-linear in *Code_Aster*

- Material is steel with Von Mises plasticity with isotropic hardening, traction curve from file

```
FSIGM = LIRE_FONCTION(UNITE=21,PARA='EPSI')

STEEL = DEFI_MATERIAU( ELAS=_F(YOUNG=210.E9,NU=0.3),
                       TRACTION=_F(SIGM=FSIGM))
```

- Steel on all mesh

```
CHMA  = AFFE_MATERIAU( MAILLAGE=MESH,
                       AFFE=_F(  TOUT='OUI',
                                 MATER='STEEL'))
```

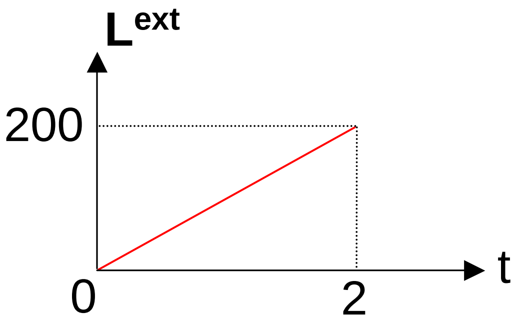
Non-linear in *Code_Aster*

- Plasticity with isotropic hardening, small strains

```
RESUN = STAT_NON_LINE(...
                CHAM_MATER = CHMA,
                COMP_INCR  = _F(
                      TOUT        ='OUI',
                      RELATION    ='VMIS_ISOT_TRAC',
                      DEFORMATION ='PETIT'),
                ...)
```

Non-linear in *Code_Aster*

- Loading : parametrized by time – Using function in **FONC_MULT**



$$\left\{L^{ext}(t,u(t))\right\} = g(t) \, . \left\{L^{ext}(u)\right\}$$

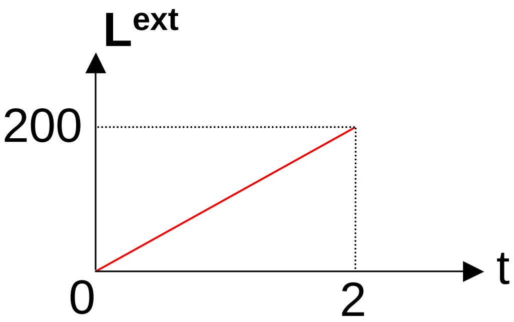
```
RAMPE = DEFI_FONCTION(PARA='INST',VALE=(0,0,2,2))
```

```
LOAD  = AFFE_CHAR_MECA(MODELE=MOD,
                       PRES_REP=_F(PRES=100.,
                                   GROUP_MA='TOTO'))
```

```
RESUN = STAT_NON_LINE(...
                      EXCIT=_F(CHARGE   =LOAD,
                               FONC_MULT=RAMPE)
                      ...)
```

Non-linear in *Code_Aster*

- Loading : parametrized by time – Using function directly in BC



$$\{L^{ext}(t,u(t))\} = g(t) \cdot \{L^{ext}(u)\}$$

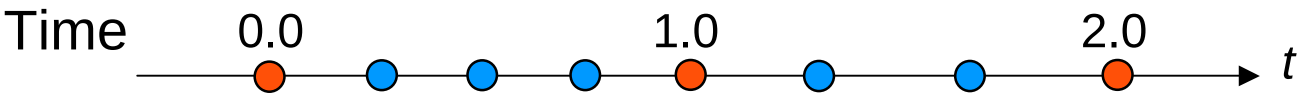
```
RAMPE = DEFI_FONCTION(PARA='INST',VALE=(0,0,2,200))
```

```
LOAD  = AFFE_CHAR_MECA_F(MODELE=MOD,
                         PRES_REP=_F(PRES=RAMPE,
                                     GROUP_MA='TOTO'))
```

```
RESUN = STAT_NON_LINE(...
                      EXCIT=_F(CHARGE   =LOAD)
                      ...)
```

Non-linear in *Code_Aster*

- Computing is parametrized by time

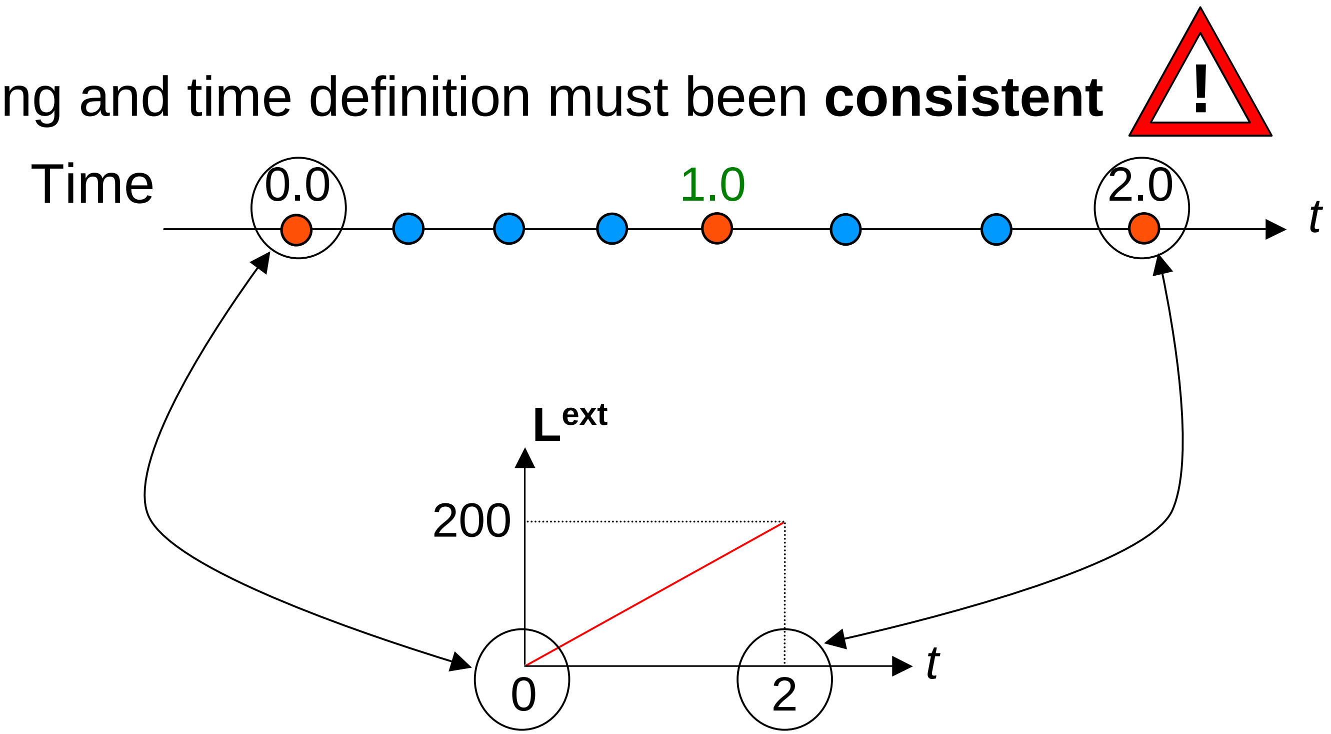



```
L_INST=DEFI_LIST_REEL( DEBUT=0.0,
                       INTERVALLE=(
                          _F(JUSQU_A=1.0,NOMBRE=3,),
                          _F(JUSQU_A=2.0,NOMBRE=2,),
                       ),)
```

```
RESUN = STAT_NON_LINE(...
                      INCREMENT=_F(LIST_INST=L_INST)
                      ...)
```

Non-linear in *Code_Aster*

- Loading and time definition must been **consistent**



- Computation on **part** of step

```
RESUN = STAT_NON_LINE(...
                             INCREMENT=_F(LIST_INST=L_INST,
                                          INST_FIN =1.0)
                             ...)
```

Non-linear in *Code_Aster – Advanced controls*

V11

- Automatic step cut : no convergence → automatic cutting

```
L_INST=DEFI_LIST_REEL( DEBUT=0.0,
                       INTERVALLE=(
                           _F(JUSQU_A=1.0,NOMBRE=3,)
                           _F(JUSQU_A=2.0,NOMBRE=2,),
                    ),)
```

```
DEFLIST = DEFI_LIST_INST(
                        DEFI_LIST = _F(
                              LIST_INST = L_INST,),
                        ECHEC     = _F(
                            EVENEMENT = 'ERREUR',
                            ACTION    = 'DECOUPE',),)
```

```
RESUN = STAT_NON_LINE(...
                      INCREMENT=_F(LIST_INST=DEFLIST)
                      ...)
```

Non-linear in *Code_Aster – Advanced controls*

V11

- Automatic step adaptation :

```
L_INST=DEFI_LIST_REEL( DEBUT=0.0,
         INTERVALLE=(_F(JUSQU_A=2.0,NOMBRE=1,),
                   ),)
```

```
DEFLIST = DEFI_LIST_INST(  DEFI_LIST = _F(
                           METHODE   = 'AUTO'
                           LIST_INST = L_INST,
                           PAS_MINI  = 1.E-6),)
```

```
RESUN = STAT_NON_LINE(...
                   INCREMENT=_F(LIST_INST=DEFLIST)
                   ...)
```

Non-linear in *Code_Aster – Advanced controls*

- Quasi-Newton method : recompute matrix every 2 Newton's iteration

```
RESUN = STAT_NON_LINE(...
                       NEWTON=_F(REAC_ITER=2,),)
```

- Quasi-Newton method : recompute matrix every 3 time's step

```
RESUN = STAT_NON_LINE(...
                       NEWTON=_F(REAC_INCR=3,),)
```

- Quasi-Newton method : using elastic matrix

```
RESUN = STAT_NON_LINE(...
                       NEWTON=_F(MATRICE='ELASTIQUE',),)
```

- Default values in *Code_Aster*

```
RESUN = STAT_NON_LINE(...
                   NEWTON=_F(   REAC_INCR=1,
                                REAC_ITER=0,
                                MATRICE='ELASTIQUE',),)
```


Non-linear in *Code_Aster – Advanced controls*

- Convergence criterion : careful when change values !

```
RESUN = STAT_NON_LINE(...
                       CONVERGENCE=_F(
                       RESI_GLOB_MAXI=1.E-6,
                       RESI_GLOB_RELA=1.E-6,),)
```

- Changing criterion convergence may produces **WRONG** results
- Maximum number of Newton's iteration

```
RESUN = STAT_NON_LINE(
                CONVERGENCE=_F(ITER_GLOB_MAXI=20,),)
```

- Elastic matrix uses thousands of Newton's iterations
- Default values in *Code_Aster*

```
RESUN = STAT_NON_LINE(...
                CONVERGENCE=_F(
                         ITER_GLOB_MAXI=10,
                         RESI_GLOB_RELA=1.E-6,),)
```

End of presentation

Is something missing or unclear in this document?

Or feeling happy to have read such a clear tutorial?

Please, we welcome any feedbacks about Code_Aster training materials.

Do not hesitate to share with us your comments on the Code_Aster forum dedicated thread.