0.1 Stresses due to the shear cross section resultants

In the presence of nonzero shear resultants, the bending moment exhibits a linear variation with the axial coordinate z in a straight beam. Based on the beam segment equilibrium we have

$$S_y = \frac{d\mathcal{M}_x}{dz}, \quad S_x = -\frac{d\mathcal{M}_y}{dz}, \tag{1}$$

as rationalized in Fig. 2, with $dz \to 0$ and $\mathcal{M}_x, \mathcal{M}_y$ differentiable with respect to z.

The linear variation of the bending-induced curvature in z causes a likewise linear variation of the pointwise axial strain; stress variation is also linear in the case of constant E_z longitudinal elastic modulus.

In particular, the differentiation with respect to z of σ_z as espressed in Eqn. **??** returns

$$\frac{d\sigma_z}{dz} = \alpha\left(x, y, E_z, \overline{EJ}_{**}\right) S_y - \beta\left(x, y, E_z, \overline{EJ}_{**}\right) S_x \tag{2}$$

since its α, β, γ factors are constant with respect to z; the bending moment derivatives are here expressed in terms of the shear resultants, as in Eqns. 1.

Figure 1 rationalizes the axial equilibrium for an elementary volume of material; we have

$$\frac{d\tau_{zx}}{dx} + \frac{d\tau_{yz}}{dy} + \frac{d\sigma_z}{dz} + q_z = 0 \tag{3}$$

where, for the specific case, the distributed volumetric load q_z is zero.

It clearly emerges from such relation that the shear stresses τ_{zx}, τ_{yz}, that were null within the uniform bending framework, are non-uniform along the section – and hence not constantly zero – in the presence of shear resultants.

A treatise on the pointwise solution of a) the equilibrium equations 3, once coupled with b) the compatibility conditions and with c) the the material elastic response, is beyond the scope of the present contribution, although it has been derived for selected cross sections in e.g. [1].

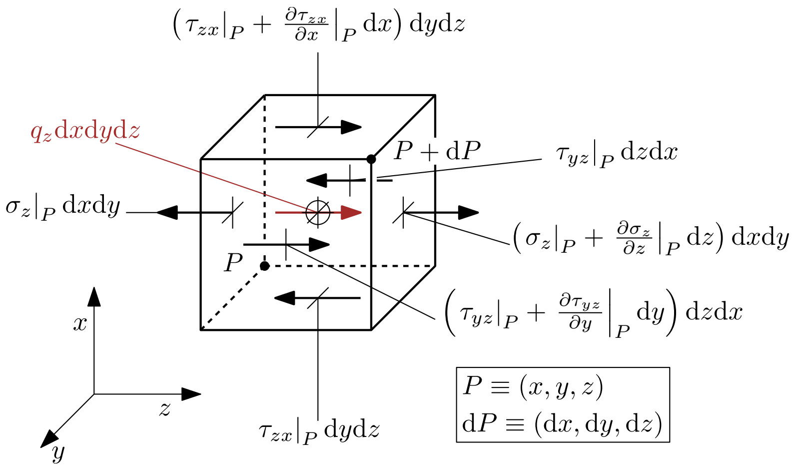

Figure 1: Equilibrium conditions with respect to the axial z translation for the infinitesimal volume extracted from the beam. In the case under scrutiny, the distributed volume action q_z is null.

0.1.1 The Jourawsky approach and its extension for a general section

The aforementioned axial equilibrium condition, whose treatise is cumbersome for the infinitesimal volume, may be more conveniently dealt with if a finite portion of the beam segment is taken into account, as in Figure 2.

A beam segment is considered whose axial extent is dz; the beam cross section is partitioned based on a (possibly curve, see Fig. 3) line that isolates an area portion A^* – and the related beam segment portion – for further scrutiny; axial equilibrium equation may then be stated for the isolated beam segment portion as follows

$$\bar{\tau}_{zi} t = \int_{A^*} \frac{d\sigma_z}{dz} dA, \tag{4}$$

where

$$\bar{\tau}_{zi} = \frac{1}{t} \int_t \tau_{zi} dr \tag{5}$$

is the average shear stress acting in the z direction along the cutting surface; i is the (locally normal) inward direction with respect to such

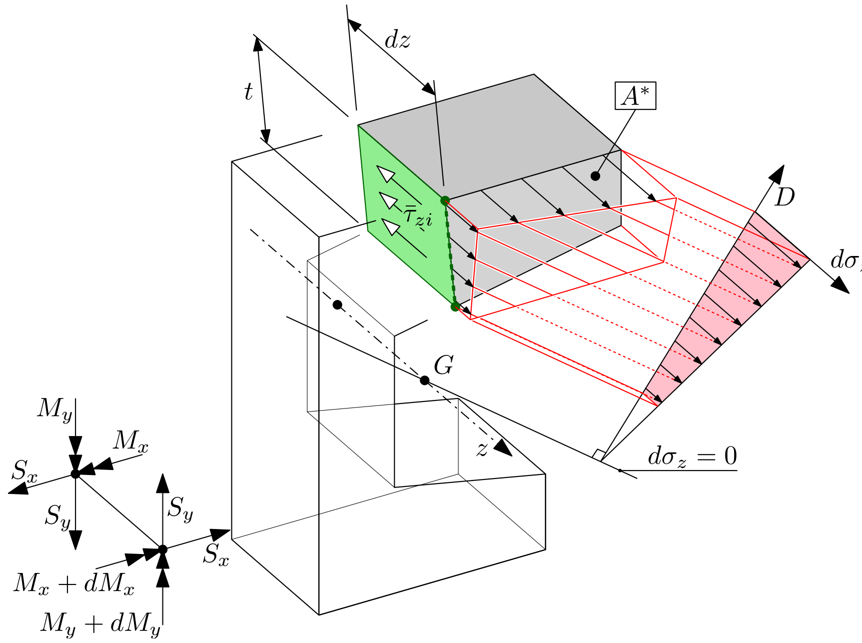

Figure 2: Equilibrium conditions for the isolated beam segment portion. It is noted that the null σ_z variation locus, $d\sigma_z = 0$, does not coincide with the bending neutral axis in general. Also, the depicted linear variation of $d\sigma_z$ with the D distance from such null $d\sigma_z$ locus does not hold in the case of non-uniform E_z modulus.

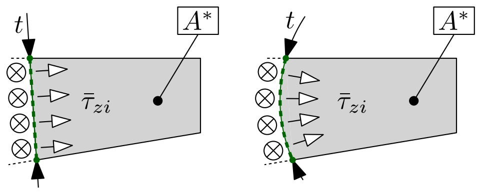

Figure 3: The curve employed for isolating the beam segment portion defines the direction of the τ_{zi} components whose average value is evaluated.

a surface. Due to the reciprocal nature of the shear stresses, the same $\bar{\tau}_{zi}$ shear stress acts along the cross sectional plane, and locally at the cutting curve itself. These shear actions are assumed positive if inward directed with respect to A^*.

The $\bar{\tau}_{zi}t$ product is named *shear flow*, and may be evaluated along a general cutting curve.

It is noted that, according to Eqn. 4, no information is provided with regard to a) the τ_{zr} shear stress that acts parallel to the cutting curve, nor b) the pointwise variation of τ_{zi} with respect of its average value $\bar{\tau}_{zi}$. If the resorting to more cumbersome calculation frameworks is not an option, those quantities are usually just neglected; an informed choice for the cutting curve is thus critical for a reliable application of the method.

In the simplified case of a) uniform material and b) local x, y axes that are principal axes of inertia (i.e. $J_{xy} = 0$), the usual formula is obtained

$$\bar{\tau}_{zi}t = \int_{A^*} \left(\frac{yS_y}{J_{xx}} + \frac{xS_x}{J_{yy}} \right) dA = \frac{\bar{y}^* A^*}{J_{xx}} S_y + \frac{\bar{x}^* A^*}{J_{yy}} S_x, \tag{6}$$

where $\bar{y}^* A^*$ and $\bar{x}^* A^*$ are the first order area moments of the A^* section portion with respect to the x and y axes, respectively[1].

0.1.2 Shear induced stresses in an open section, thin walled beam

In the case of thin walled profiles, the integral along the isolated area in Eqn. 4 may be performed with respect to the arclength coordinate alone; the value the $d\sigma_z/dz$ integrand assumes at the wall midplane is supposed representative of its integral average along the wall thickness, thus obtaining

$$\bar{\tau}_{zi}t = q_{zi} = \int_0^s \int_{-t/2}^{t/2} \frac{d\sigma_z}{dz} dr d\varsigma \approx \int_0^s \left. \frac{d\sigma_z}{dz} \right|_{r=0} t d\varsigma. \tag{7}$$

Such assumed equivalence strictly holds for a) straight wall segments[2] and b) a linear variation of the integrand along the wall, a

[1] According to the employed notation, $(\bar{x}^*, \bar{y}^*)$ are the centre of gravity coordinates for the A^* area.

[2] i.e. the Jacobian of the $(s, r) \mapsto (x, y)$ mapping is constant with r.

condition, the latter, that holds if the material properties are homogeneous with respect to the wall midplane[3]; in the more general case, the error incurred by this approach vanishes with vanishing thickness for what concerns assumption a), whereas if the material is inhomogeneous, through-thickness averaged $\bar{E}_z, \bar{G}_{zi}$ moduli may be employed in place of their pointwise counterpart.

If a thin walled section segment is considered such that it is not possible to infer that the interfacial shear stress is zero at at least one of its extremities, a further term needs to be considered for the equilibrium, thus obtaining

$$\bar{\tau}_{zi}(s)t(s) = q(s) = \int_a^s \frac{d\sigma_z}{dz} t d\varsigma + \underbrace{\bar{\tau}_{zi}(a)t(a)}_{q_A}. \tag{8}$$

In the case of open thin walled profiles, however, such a choice for the isolated section portion is suboptimal, unless the q_A term is known.

0.1.3 Shear induced stresses in an closed section, thin walled beam

In the case of a closed thin walled, generally asymmetric section, the search for a point along the wall at which the shear flow may be assumed zero is normally not viable, and the employment of Eq. 8 in place of the simpler Eq. 7 is unavoidable.

In this case, a parametric value for the $\bar{\tau}_{iz}$ shear stress is assumed for a set of points along the cross section midcurve – one for each elementary closed loop[4] if the points are non-redundantly chosen[5].

In the multicellular cross section example shown in Figure 4, two elementary loops are detected; shear flows at the A, B points are parametrically defined as $\tau_A t_A$ and $\tau_B t_B$, respectively.

The $\tau(s)$ shear stress at each point along the profile wall may then be determined based on Eqn. 8 as a function a) of the shear resultant

[3]a linear $d\epsilon_z/dz$ axial strain variation is in fact associated to the curvature variation in z, and not an axial stress variation;

[4]i.e. a closed loop not enclosing any other closed loop.

[5]Redundancy may be pointed out by ideally cutting the cross section at these points: if a monolithic open cross section is obtained, the point choice is not redundant; if a portion of the section is completely isolated, and a loop remains closed, the location of these points causes redundancy.

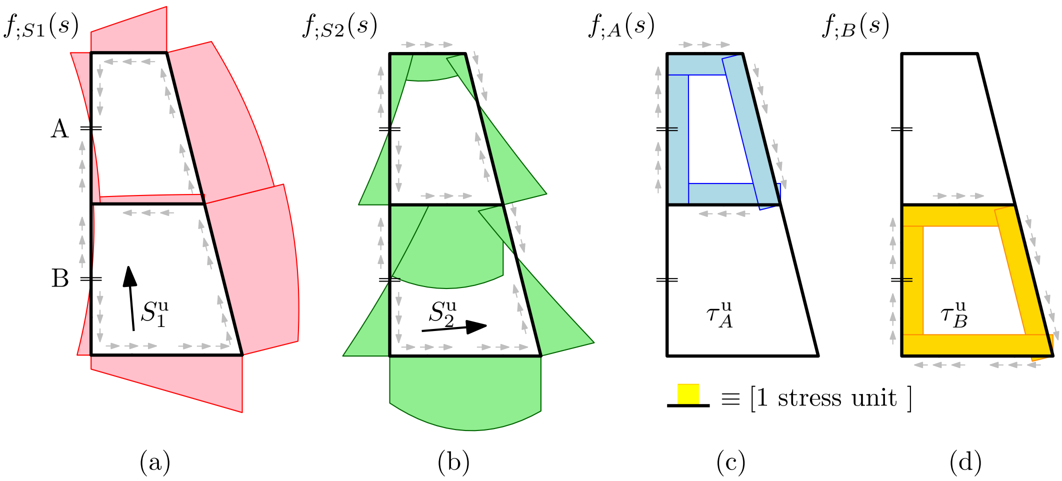

Figure 4: Contributions to the $\tau_{zi}(s)$ shear stress along the profile walls associated to a) a unit shear force component S_1^{u} applied along the first principal axis of inertia, whose magnitude equals the product of the cross section area and the unit stress, b) an analogous shear force component S_2^{u} aligned with the second principal axis of inertia, c) a unit shear stress τ_A^{u} applied at the opposite fictitious cut surfaces at A, and d) a unit shear stress τ_B^{u} applied at the opposite fictitious cut surfaces at B. Profile wall thickness is constant in the presented example, thus producing a continuous shear stress diagram, whereas continuity is rather aa unit shear stress τ_A^{u} applied at the opposite fictitious cut surfaces at a property of the shear flow.

components S_x and S_y, and b) of the parametrically defined shear stresses at the A,B points.

Due to the assumed linear response for the profile, superposition principle may be employed in isolating the four elementary contributions to the shear stress flow along the section.

The first two elementary contributions $f_{;Sx}(s)$ and $f_{;Sy}(s)$ are respectively due to the action alone of the x and y shear force components, whose magnitudes S_x^{u} and S_y^{u} is assumed equal the product of the stress unit (e.g. 1 MPa) and of the cross sectional area. Those forces are assumed to act in the ideal absence of shear flow at points where the latter is assumed as a parameter (points A and B in Figure 4).

Since the condition of zero shear flow is stress-compatible with an opening in the closed section loop, the cross section may be idealized as severed at the assumed shear flow points, and hence open. The equilibrium-based solution procedure derived for the open thin-walled section may hence be profitably applied.

A family of further elementary contributions, one for each of the assumed shear stress points, may be derived by imposing zero parametric shear flow at all the points but the one under scrutiny, and in the absence of externally applied shear resultants. The elastic problem may be rationalized as an open – initially closed, then ideally severed – thin walled profile, that is loaded by an internal constraint action whose magnitude is unity in terms of stresses. Equilibrium considerations reduce to the conservation of the shear flow due to the absence of $d\sigma_z/dz$ differential axial stress, as in the case of a closed profile under torsion discussed below.

Figures 4 (a) and (b) show the shear stress contributions $f_{;S1}(s)$ and $f_{;S2}(s)$ induced in the ideally opened (i.e. zero redundant shear flows at the A,B points) multicellar profile by the first and the second shear force components, respectively; due to the author distraction, such figure refers to shear components aligned with the principal directions of bending stiffness, and not to the usual x,y axes.

Figures 4 (c) and (d) show the shear stress contributions $f_{;A}(s)$ and $f_{;B}(s)$ associated to unity values for the parametric shear flows at the A, B segmentation points, respectively.

The cumulative shear stress distribution for the section in Figure 4 is

$$\tau(s) = \frac{S_1}{\mathcal{A}} f_{;S1}(s) + \frac{S_2}{\mathcal{A}} f_{;S2}(s) + \tau_A f_{;A}(s) + \tau_B f_{;B}(s) \tag{9}$$

where s is a suitable arclength coordinate.

The associated elastic potential energy may then be integrated over a Δz beam axial portion, thus obtaining

$$\Delta U = \int_s \frac{\tau^2}{2G_{sz}} t \Delta z ds \tag{10}$$

According to the Castigliano second theorem, the ΔU derivative with respect to the $\bar{\tau}_i$ assumed shear stress value at the i-th segmentation point equates the generalized displacement with respect to which the internal constraint reaction works, i.e. the $t\Delta z\bar{\delta}_i$ integral of the relative longitudinal displacement between the cut surfaces; we hence have

$$\frac{\partial \Delta U}{\partial \bar{\tau}_i} = \bar{\delta}_i t \Delta z \tag{11}$$

The $\bar{\delta}_i$ symbol refers to the average value along the $t\Delta z$ area of such axial relative displacement.

Material continuity requires zero $\bar{\delta}_i$ value at each segmentation point, thus defining a set of equations, one for each $\bar{\tau}_i$ unknown parameter, whose solution leads to the definition of the actual shear stress distribution along the closed wall profile.

0.2 Shear stresses due to the St. Venant torsion

The classical solution for the rectilinear beam subject to uniform torsion predicts a displacement field that is composed by the superposition of a) a rigid, in-plane[6] cross section rotation about the shear centre, named twist, of uniform axial rate, and b) an out-of-plane *warping* displacement that is uniform in the axial direction, whereas it varies within the section; such warping displacement is zero in the case of axisymmetric sections only (e.g. solid and hollow circular cross sections).

The in-plane stress components σ_x, σ_y, τ_{xy} are assumed zero, along with the normal stress σ_z. The motion is internally restricted only due to the nonzero out-of-plane shear stresses τ_{yz} and τ_{zx}, that develop as an elastic reaction to the associated strain components.

A more in-depth treatise of the topic involves the solution of an plane, inhomogeneous Laplace differential equation with essential conditions imposed at the cross section boundary, which is beyond the scope of the present contribution.

However, in the case of open- and closed- section, thin walled beams, simplified solutions are available based on the assumptions that a) the out-of-plane shear stresses are locally aligned to the wall midsurface - i.e. $\tau_{zr} = 0$ leaving τ_{zs} as the only nonzero stress component[7], and b) the residual τ_{zs} shear component is either constant by moving through the wall thickness (closed section case), or it linearly varies with the through-thickness coordinate r.

0.2.1 Solid section beam

TODO.

0.2.2 Closed section, thin walled beam

The τ_{sz} component is assumed uniform along the wall thickness, or, equivalently, its deviation from the average value is neglected in calculations.

[6]the rotation vector is actually normal to the cross sectional plane; the *in-plane* motion characterization refers to the associated displacement field.

[7]Here, the notation introduced in paragraph XXX for the thin walled section is employed.

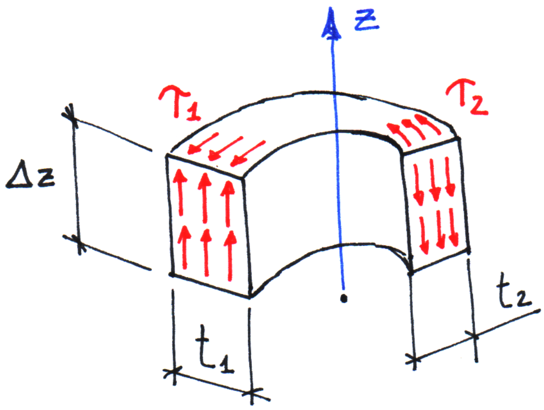

Figure 5: Axial equilibrium for a portion of profile wall, in the case of a closed, thin-walled profile subject to torsion.

In the case the material is non-uniform across the thickness, the γ_{sz} shear strain is assumed uniform, whereas the τ_{sz} varies with the varying G_{sz} shear modulus.

In the absence of σ_z, the axial equilibrium of a portion of beam segment dictates that the shear flow $t\tau$ remains constant along the wall, i.e.

$$t_1\tau_1 = t_2\tau_2$$

as depicted in Figure 5.

By skipping some further interesting observations (TODO) we may just introduce the Bredt formula for the cross-section torsional stiffness

$$K_t = \frac{4A^2}{\oint \frac{1}{t} dl} \tag{12}$$

which is valid for single-celled, closed thin wall sections.

The peak stress is located at thinnest point along the wall, and equals

$$\tau_{\max} = \frac{M_t}{2t_{\min}A} \tag{13}$$

.

Multi-celled beam profile? TODO.

0.2.3 Open section, thin walled beam

The shear strain component γ_{zs} is assumed linearly varying across the thickness; if the G_{sz} shear modulus is assumed uniform, such linear variation characterizes the τ_{zs} stress components too.

The average value along the thickness of the τ_{zs} stress component is zero, as zero is the shear flow as defined in the previous paragraph.

For thin enough open sections of uniform and isotropic material we have

$$K_T \approx \frac{1}{3} \int_0^l t^3(s) ds \tag{14}$$

If the thin-walled cross section may be described as a sequence of constant thickness wall segments, the simplified formula

$$K_T \approx \frac{1}{3} \sum_i l_i t_i^3 \tag{15}$$

is obtained where t_i and l_i are respectively the length and the thickness of each segment.

The peak value for the τ_{zs} stress component is observed in correspondence to thickest wall section point and it equates

$$\tau_{\max} = \frac{M_t t_{\max}}{K_T} \tag{16}$$

By applying the reported formulas to a rectangular section whose span length is ten times the wall thickness, the torsional stiffness is overestimated by slightly less than 7%; a similar relative error is reported in terms of shear stress underestimation.

Bibliography

[1] A. E. H. Love, *A treatise on the mathematical theory of elasticity.* Cambridge university press, 2013.