0.1 Basic theory of plates

P displacement components as a function of the Q reference point motion.

$$u_P = u + z\,(1+\breve{\epsilon}_z)\sin\varphi \tag{1}$$
$$v_P = v - z\,(1+\breve{\epsilon}_z)\sin\theta \tag{2}$$
$$w_P = w + z\,((1+\breve{\epsilon}_z)\cos(\varphi)\cos(\theta) - 1) \tag{3}$$

$$\breve{\epsilon}(z) = \frac{1}{z}\int_0^z \epsilon_z d\varsigma \tag{4}$$
$$= \frac{1}{z}\int_0^z -\frac{\nu}{1-\nu}\left(\epsilon_x + \epsilon_y\right) d\varsigma \tag{5}$$

P displacement components as a function of the Q reference point motion, linarized with respect to the small rotations and small strain hypotheses.

$$u_P = u + z\varphi \tag{6}$$
$$v_P = v - z\theta \tag{7}$$
$$w_P = w \tag{8}$$

Relation between the normal displacement x, y gradient (i.e. the deformed plate slope), the rotations and the out-of-plane, interlaminar, averaged shear strain components.

$$\frac{\partial w}{\partial x} = \bar{\gamma}_{zx} - \varphi \tag{9}$$
$$\frac{\partial w}{\partial y} = \bar{\gamma}_{yz} + \theta \tag{10}$$

Strains at point P.

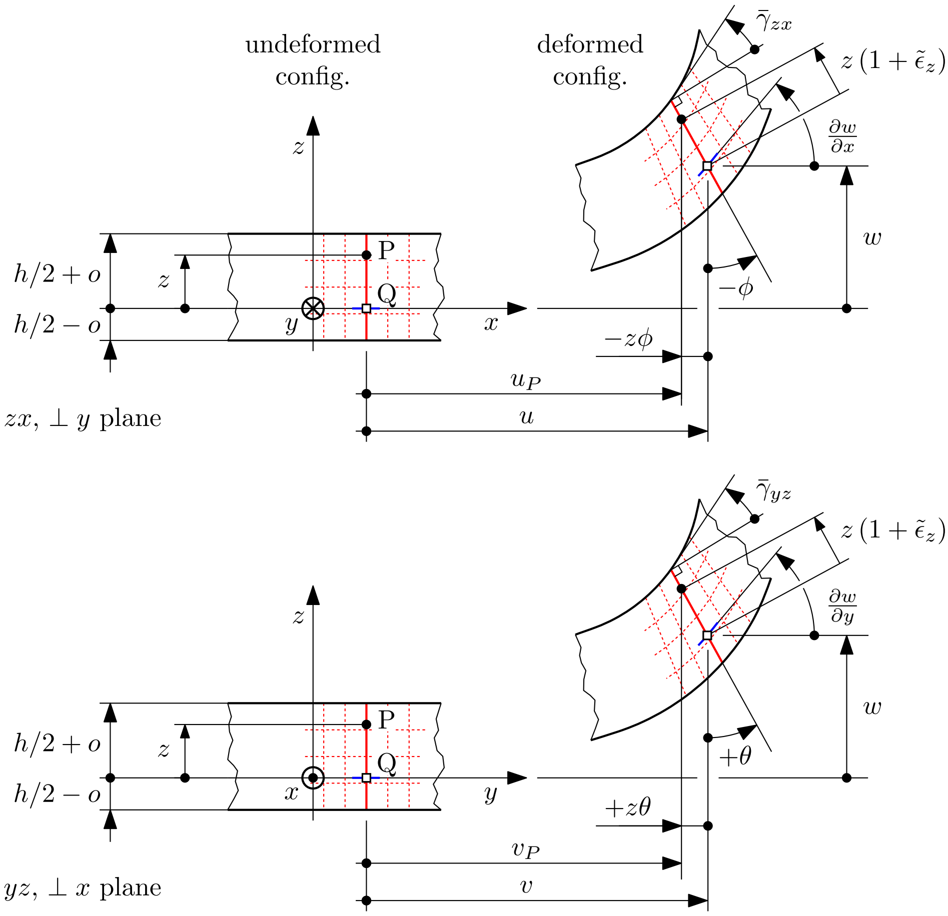

Figure 1: Relevant dimensions for describing the deformable plate kinematics.

$$\epsilon_x = \frac{\partial u_P}{\partial x} = \frac{\partial u}{\partial x} + z\frac{\partial \varphi}{\partial x} \tag{11}$$

$$\epsilon_y = \frac{\partial v_P}{\partial y} = \frac{\partial v}{\partial y} - z\frac{\partial \theta}{\partial y} \tag{12}$$

$$\gamma_{xy} = \frac{\partial u_P}{\partial y} + \frac{\partial v_P}{\partial x} \tag{13}$$

$$= \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + z\left(+\frac{\partial \varphi}{\partial y} - \frac{\partial \theta}{\partial x}\right) \tag{14}$$

Generalized plate strains: membrane strains.

$$\underline{\bar{\epsilon}} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} \bar{\epsilon}_x \\ \bar{\epsilon}_y \\ \bar{\gamma}_{xy} \end{pmatrix} \tag{15}$$

Generalized plate strains: curvatures.

$$\underline{\kappa} = \begin{pmatrix} +\frac{\partial \varphi}{\partial x} \\ -\frac{\partial \theta}{\partial y} \\ +\frac{\partial \varphi}{\partial y} - \frac{\partial \theta}{\partial x} \end{pmatrix} = \begin{pmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix} \tag{16}$$

Compact form for the strain components at P.

$$\underline{\epsilon} = \underline{\bar{\epsilon}} + z\,\underline{\kappa} \tag{17}$$

Hook law for an isotropic material, under plane stress conditions.

$$\underline{\underline{D}} = \frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \tag{18}$$

Normal components for stress and strain, the latter for the isotropic material case only.

$$\sigma_z = 0 \tag{19}$$

$$\epsilon_z = -\frac{\nu}{1-\nu}\left(\epsilon_x + \epsilon_y\right) \tag{20}$$

Stresses at P.

$$\underline{\sigma} = \underline{\underline{D}}\,\underline{\epsilon} = \underline{\underline{D}}\,\underline{\bar{\epsilon}} + z\,\underline{\underline{D}}\,\underline{\kappa} \tag{21}$$

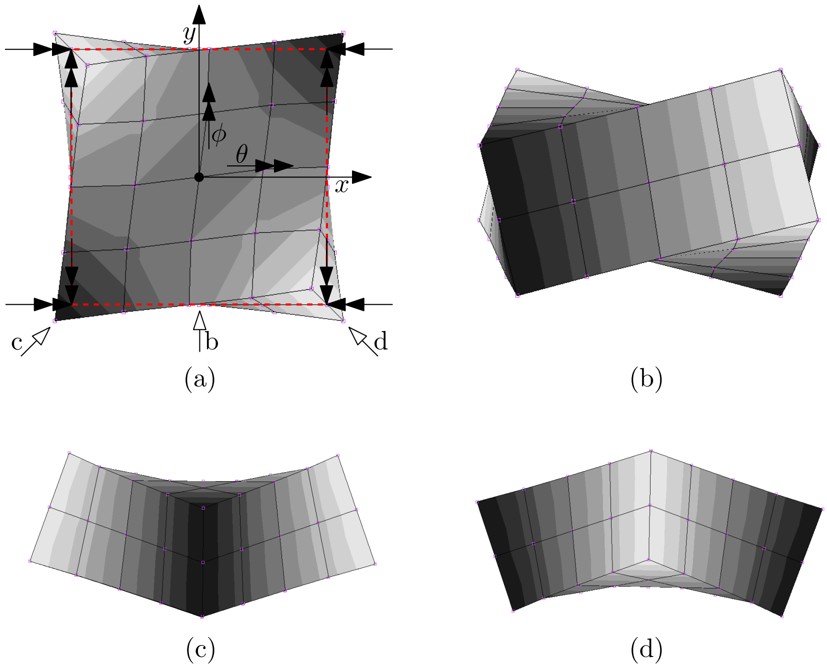

Figure 2: Positive κ_{xy} torsional curvature for the plate element. Subfigure (a) shows the positive γ_{xy} shear strain at the upper surface, the (in-plane) undeformed midsurface, and the negative γ_{xy} at the lower surface; the point of sight related to subfigures (b) to (d) are also evidenced. θ and φ rotation components decrease with x and increase with y, respectively, thus leading to positive κ_{xy} contributions. As shown in subfigures (c) and (d), the torsional curvature of subfigure (b) evolves into two anticlastic bending curvatures if the reference system is aligned with the square plate element diagonals, and hence rotated by 45° with respect to z.

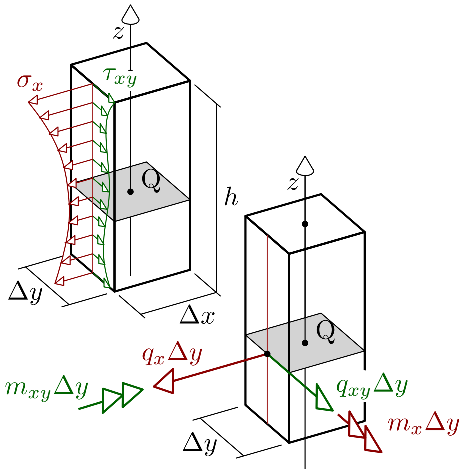

Figure 3: XXX

Membrane (direct and shear) stress resultants (stress flows).

$$\underline{q} = \begin{pmatrix} q_x \\ q_y \\ q_{xy} \end{pmatrix} = \int_h \underline{\sigma}\, dz \tag{22}$$

$$= \underbrace{\int_h \underline{\underline{D}}\, dz}_{\underline{\underline{A}}}\ \underline{\bar{\epsilon}} + \underbrace{\int_h \underline{\underline{D}}\, z dz}_{\underline{\underline{B}}}\ \underline{\kappa} \tag{23}$$

Bending and torsional moment stress resultants (moment flows).

$$\underline{m} = \begin{pmatrix} m_x \\ m_y \\ m_{xy} \end{pmatrix} = \int_h \underline{\sigma}\, dz \tag{24}$$

$$= \underbrace{\int_h \underline{\underline{D}}\, z dz}_{\underline{\underline{B}} \equiv \underline{\underline{B}}^{\mathrm{T}}}\ \underline{\bar{\epsilon}} + \underbrace{\int_h \underline{\underline{D}}\, z^2 dz}_{\underline{\underline{C}}}\ \underline{\kappa} \tag{25}$$

Cumulative generalized strain - stress relations for the plate (or for the laminate)

$$\begin{pmatrix} \underline{q} \\ \underline{m} \end{pmatrix} = \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{B}}^{\mathrm{T}} & \underline{\underline{C}} \end{pmatrix} \begin{pmatrix} \underline{\bar{\epsilon}} \\ \underline{\kappa} \end{pmatrix} \tag{26}$$

Hook law for the orthotropic material in plane stress conditions, with respect to principal axes of orthotropy;

$$\underline{\underline{D}}_{123} = \begin{pmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{pmatrix} \tag{27}$$

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{pmatrix} = \underline{\underline{T}}_1 \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \qquad \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{pmatrix} = \underline{\underline{T}}_2 \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} \tag{28}$$

where

$$\underline{\underline{T}}_1 = \begin{pmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{pmatrix} \tag{29}$$

$$\underline{\underline{T}}_2 = \begin{pmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2-n^2 \end{pmatrix} \tag{30}$$

α is the angle between 1 and x;

$$m = \cos(\alpha) \qquad n = \sin(\alpha) \tag{31}$$

The inverse transformations may be obtained based on the relations

$$\underline{\underline{T}}_1^{-1}(+\alpha) = \underline{\underline{T}}_1(-\alpha) \qquad \underline{\underline{T}}_2^{-1}(+\alpha) = \underline{\underline{T}}_2(-\alpha) \tag{32}$$

Finally

$$\underline{\sigma} = \underline{\underline{D}}\,\underline{\epsilon} \qquad \underline{\underline{D}} \equiv \underline{\underline{D}}_{xyz} = \underline{\underline{T}}_1^{-1}\underline{\underline{D}}_{123}\underline{\underline{T}}_2 \tag{33}$$

Notes:

- Midplane is ill-defined if the material distribution is not symmetric; the geometric midplane (i.e. the one obtained by ignoring the material distribution) exhibits no relevant properties in general. Its definition is nevertheless pretty straighforward.

- If the unsimmetric laminate is composed by isotropic layers, a reference plane may be obtained for which the $\underline{\underline{B}}$ membrane-to-bending coupling matrix vanishes; a similar condition may not be verified in the presence of orthotropic layers.

- Thermally induced distortion is not self-compensated in an unsymmetric laminate even if the temperature is held constant through the thickness.

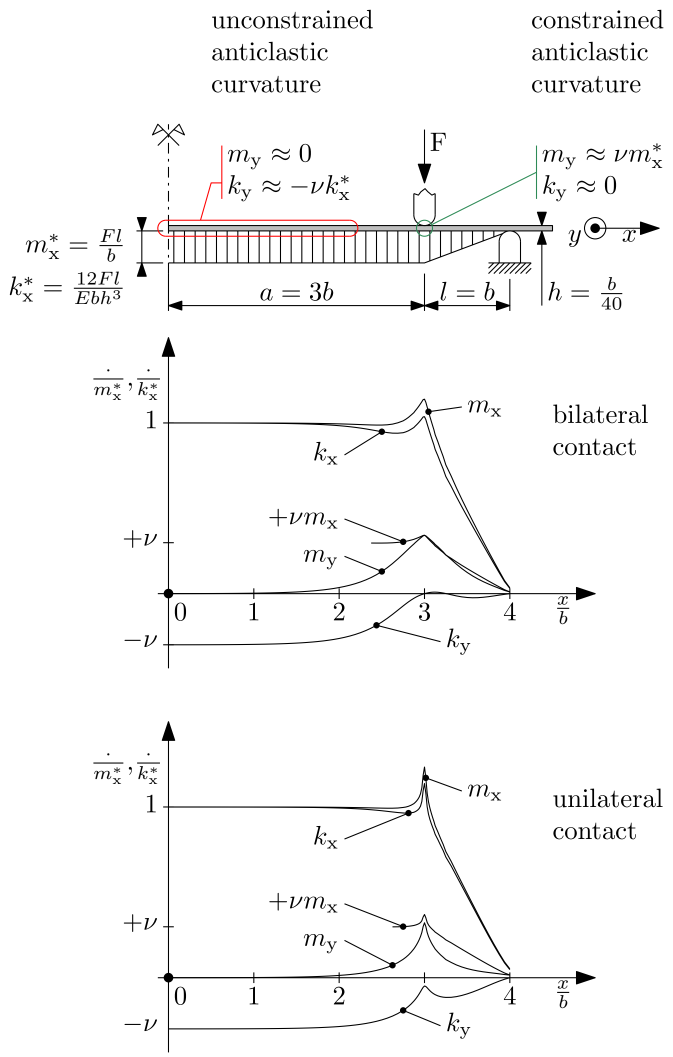

Figure 4: The *not-so-simple* four point bending case.