0.0.1 Linearized pre-buckling analysis

A few notes.

According to the linearized pre-buckling analysis, the structure is considered in an oxymoronic configuration which is both *pre-stressed* and *undeformed*.

The $\underline{\sigma}^0$ *pre-stress* condition is evaluated through a linear preliminary analysis of the structure subject to a set of applied loads, and potentially inhomogeneous constraints; both the preload and the associate stress field may be scaled by a common λ amplification factor, and the structure behaviour is parametrically examined with varying λ.

The displacement and rotation fields associated this preliminary analysis are not however retained in the subsequent step, in contrast to the pre-stress ; such looseness is commonly justified based on the assumed smallness of such deflections.

For each element of the structure, the stiffness matrix is derived by a) taking into account the contribution of the $\underline{\sigma}^0$ pre-stress to the internal virtual work, and b) by employing a second order, nonlinear, *large rotation* formulation for the $\underline{\underline{\mathrm{B}}}$ matrix that derives the strain tensor from nodal Degree of Freedom (DOF)s. Details are here omitted[1].

The resulting element stiffness matrix is obtained as the sum of two distinct contributions; the first contribution $\underline{\underline{\mathrm{K}}}_{\mathrm{ej}}^{\mathrm{m}}$ is named *material* stiffness matrix and, in the absence of large element reorientation in space, it coincides with the customary definition of element stiffness matrix. The second contribution $\underline{\underline{\mathrm{K}}}_{\mathrm{ej}}^{\mathrm{g}}$ is named *geometric* stiffness matrix and it embodies the corrective terms due to the interaction of the pre-stress with the rotations; such term is invariant with the material properties, and it scales with the pre-stress itself, i.e. with the λ amplification factor. This second contribution embodies the *stress stiffening* and *stress softening* effects.

Both the two terms are obtained by relying on the initial coordinates of the element nodes, thus effectively neglecting the preload-induced deflections.

The elemental material and geometric stiffness matrix are then assembled into their global counterparts, and contraints are applied that

[1] see e.g. reference [1]

are consistent[2] with the ones employed in deriving the pre-stress.

The following relation is thus obtained in the neighborhood of a λ-scaled, pre-stressed configuration

$$\left(\underline{\underline{\mathrm{K}}}^{\,\mathrm{m}} + \lambda \, \underline{\underline{\mathrm{K}}}^{\,\mathrm{g}} \right) \delta \, \underline{\mathrm{d}} \; = \delta \, \underline{\mathrm{F}} \tag{1}$$

that relates a small variation in the externally applied actions $\delta \, \underline{\mathrm{F}}$ with the required adjustments in the structure configuration $\delta \, \underline{\mathrm{d}}$ for the sake of equilibrium; the cumulative $\underline{\underline{\mathrm{K}}}^{\,\mathrm{m}} + \lambda \, \underline{\underline{\mathrm{K}}}^{\,\mathrm{g}}$ term is named *tangent* stiffness matrix upon its role in locally orienting the equilibrium path.

Of a particular interest is the case of a nonzero variation in configuration for which equilibrium is preserved in the absence of external load variation; such condition is a prerequisite for a bifurcation of the equilibrium path. We have in particular an homogenous system of equations

$$\left(\underline{\underline{\mathrm{K}}}^{\,\mathrm{m}} + \lambda_i \, \underline{\underline{\mathrm{K}}}^{\,\mathrm{g}} \right) \delta \, \hat{\underline{\mathrm{d}}}_{\,i} = 0 \tag{2}$$

whose nontrivial solutions are in form of generalized[3] eigenpairs $(\lambda_i, \delta \, \hat{\underline{\mathrm{d}}}_{\,i})$, with λ_i values that zero the determinant of the tangent stiffness matrix, and are hence named *critical* pre-stress (or preload, or load) amplification factor.

In correspondence of critical λ_i values, the elastic reactions are unable to restrain an arbitrary scaled $\delta \, \hat{\underline{\mathrm{d}}}_{\,i}$ perturbation of the structure configuration, and the related variation in stress/strain values, thus obtaining a *indifferent equilibrium* condition.

XXX

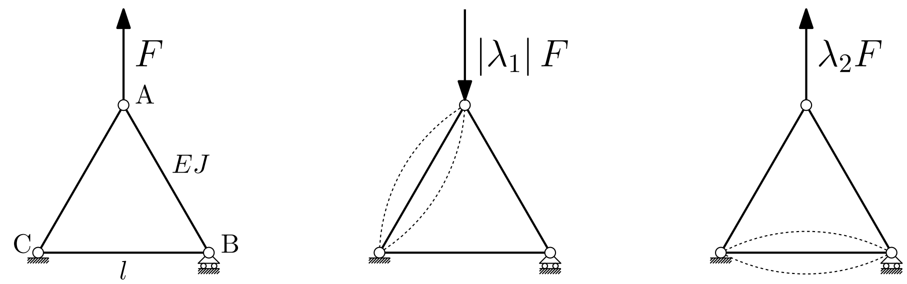
[2] not stricty equal in theory, since some variations are allowed with respect in particular positioning and symmetry constraints. FE packages may however limit such theoretically allowed redefinition of constraints.

[3] an equivalent, standard

$$\left(\underline{\underline{\mathrm{A}}} - \eta_i \, \underline{\underline{\mathrm{I}}} \right) \, \underline{\mathrm{v}}_{\,i} = 0$$

eigenproblem may be defined with

$$\underline{\underline{\mathrm{A}}} = \left[\underline{\underline{\mathrm{K}}}^{\,\mathrm{m}} \right]^{-1} \underline{\underline{\mathrm{K}}}^{\,\mathrm{g}}, \quad \lambda_i = -1/\eta_i, \quad \underline{\mathrm{v}}_{\,i} = \delta \, \hat{\underline{\mathrm{d}}}_{\,i}.$$



$N_{AB} = N_{CA} = +\frac{F}{\sqrt{3}}$ $\quad$ $\lambda_1 = -\frac{\pi^2 EJ}{l^2}\frac{\sqrt{3}}{F}$ $\quad$ $\lambda_2 = +\frac{\pi^2 EJ}{l^2}\frac{2\sqrt{3}}{F}$

$N_{BC} = -\frac{F}{2\sqrt{3}}$ $\quad$ moltepl. 2

$N_{\mathrm{crit},i} = -\frac{\pi^2 i^2 EJ}{l^2}$ $\quad$ minimum λ_i in absolute value $\quad$ minimum $\lambda_i > 0$

Figure 1: Rationalization of the minimum critical amplification factor in modulus vs. minimum critical positivi amplification factor problem.

Bibliography

[1] J. Oden, “Calculation of geometric stiffness matrices for complex structures.,” *AIAA Journal*, vol. 4, no. 8, pp. 1480–1482, 1966.