0.1 Joining elements into structures.

0.1.1 Displacement and rotation field continuity

Displacement and rotation fields are continuous at the isoparametric quadrilateral inter-element interfaces; they are in fact continuous at nodes since the associated nodal Degree of Freedom (DOF)s are shared by adjacent elements, and the field interpolations that occur within each quadrilateral domain a) they both reduce to the same linear relation along the shared edge, and b) they are performed in the absence of any contributions related to unshared nodes.

0.1.2 Expressing the element stiffness matrix in terms of global DOFs

As seen in Par. **??**, the stiffness matrix of each j-th element defines the elastic relation between the associated generalized forces and displacements, i.e.

$$\underline{\mathrm{F}}_{\mathrm{e}j} = \underline{\underline{\mathrm{K}}}_{\mathrm{e}j}\, \underline{\mathrm{d}}_{\mathrm{e}j} \tag{1}$$

where the DOFs definition is local with respect to the element under scrutiny.

In order to investigate the mutual interaction between elements in a structure, a common set of *global* DOFs is required; in particular, generalized displacement DOFs are defined at each l-th global node, i.e., for nodes interacting with the shell element formulation under scrutiny,

$$\underline{\mathrm{d}}_{\mathrm{g}l} = \begin{bmatrix} u_{\mathrm{g}l} \\ v_{\mathrm{g}l} \\ w_{\mathrm{g}l} \\ \theta_{\mathrm{g}l} \\ \varphi_{\mathrm{g}l} \\ \psi_{\mathrm{g}l} \end{bmatrix}. \tag{2}$$

The global reference system $\mathrm{O}XYZ$ is typically employed in projecting nodal vector components. However, each l-th global node may be supplied with a specific reference system, whose unit vectors are $\hat{\imath}_{\mathrm{g}l}, \hat{\jmath}_{\mathrm{g}l}, \hat{k}_{\mathrm{g}l}$, thus permitting the employment of non uniformly aligned (e.g. cylindrical) global reference systems.

Those nodal degrees of freedom may be collected in a global DOFs vector

$$\underline{\mathrm{d}}_{\mathrm{g}}^{\top} = \begin{bmatrix} \underline{\mathrm{d}}_{\mathrm{g}1}^{\top} & \underline{\mathrm{d}}_{\mathrm{g}2}^{\top} & \dots & \underline{\mathrm{d}}_{\mathrm{g}l}^{\top} & \dots & \underline{\mathrm{d}}_{\mathrm{g}n}^{\top} \end{bmatrix} \tag{3}$$

that parametrically defines any deformed configuration of the structure.

Analogously, a global, external (generalized[1]) forces vector may be defined, that assumes the form

$$\underline{\mathrm{F}}_{\mathrm{g}}^{\top} = \begin{bmatrix} \underline{\mathrm{F}}_{\mathrm{g}1}^{\top} & \underline{\mathrm{F}}_{\mathrm{g}2}^{\top} & \dots & \underline{\mathrm{F}}_{\mathrm{g}l}^{\top} & \dots & \underline{\mathrm{F}}_{\mathrm{g}n}^{\top} \end{bmatrix}; \tag{4}$$

since external constraints are expected to be applied to the structure DOFs, the following vector of reaction forces

$$\underline{\mathrm{R}}_{\mathrm{g}}^{\top} = \begin{bmatrix} \underline{\mathrm{R}}_{\mathrm{g}1}^{\top} & \underline{\mathrm{R}}_{\mathrm{g}2}^{\top} & \dots & \underline{\mathrm{R}}_{\mathrm{g}l}^{\top} & \dots & \underline{\mathrm{R}}_{\mathrm{g}n}^{\top} \end{bmatrix} \tag{5}$$

is introduced, along with the reaction force vector of the *internal* kinematic constraints, named *tying forces*

$$\underline{\mathrm{T}}_{\mathrm{g}}^{\top} = \begin{bmatrix} \underline{\mathrm{T}}_{\mathrm{g}1}^{\top} & \underline{\mathrm{T}}_{\mathrm{g}2}^{\top} & \dots & \underline{\mathrm{T}}_{\mathrm{g}l}^{\top} & \dots & \underline{\mathrm{T}}_{\mathrm{g}n}^{\top} \end{bmatrix}. \tag{6}$$

The simple four element, roof-like structure of Fig. 1 is employed in the following to discuss the procedure that derives the elastic response characterization for the structure from its elemental counterparts.

The structure comprises nine nodes, whose location in space is defined according to a global reference system $\mathrm{O}XYZ$, see Table 1.

The structure is composed by four, identical, four noded isoparametric shell elements, whose formulation is described in the preceding section **??**.

A grayscale, normalized representation of the element stiffness matrix is shown in Figure 2, where the white to black colormap spans from zero to the maximum in absolute value term.

The mapping between local, element based node numbering and the global node numbering is reported in the connectivity Table 2.

Such i) local to global node numbering mapping, together with ii) the change in reference system mentioned above, defines a set of elemental DOF mapping matrices, $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$, one each j-th element. Such matrices are defined as follows: the i-th row the $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$ matrix contains

[1] Unless otherwise specified, the *displacement* and *force* terms refer to the DOFs, and the suitable actions that perform work with their variation, respectively. They are in fact *generalized* forces and displacements.

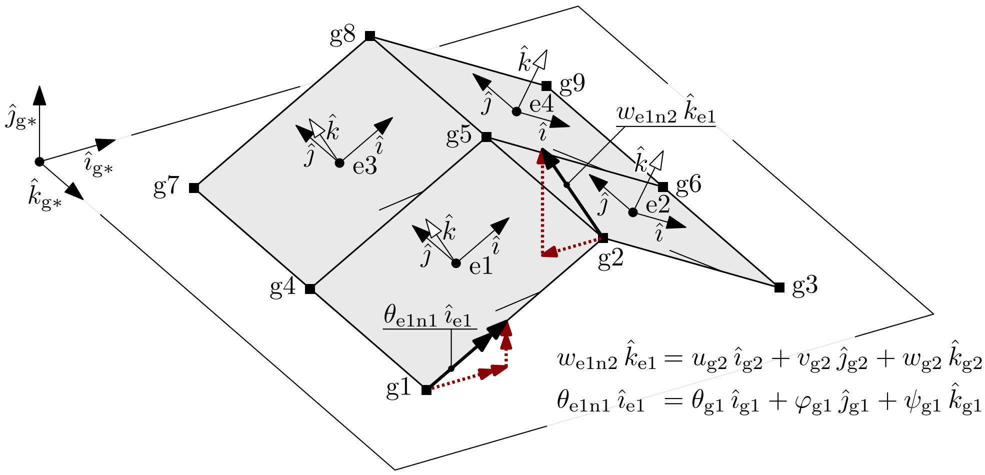

Figure 1: A simple four-element, roof-like structure employed in discussing the assembly procedures. The elements are square, thick plates whose angle with respect to the global XY plane is 30°

node	X	Y	Z
g1	$-lc$	0	$+l$
g2	0	$+ls$	$+l$
g3	$+lc$	0	$+l$
g4	$-lc$	0	0
g5	0	$+ls$	0
g6	$+lc$	0	0
g7	$-lc$	0	$-l$
g8	0	$+ls$	$-l$
g9	$+lc$	0	$-l$

Table 1: Nodal coordinates for the roof-like structure of Figure 1. l is the element side length, $c = \cos 30°$ and $s = \sin 30°$

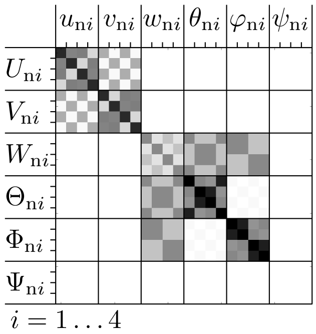

Figure 2: A representation of the stiffness matrix terms for each element in the example structure; the term magnitude is represented through a linear grayscale, spanning from zero (white) to the peak value (black).

	n1	n2	n3	n4
e1	g1	g2	g5	g4
e2	g2	g3	g6	g5
e3	g4	g5	g8	g7
e4	g5	g6	g9	g8

Table 2: Element connectivity for the roof-like structure of Figure 1. As an example, the node described by the local numbering e3n2 is mapped to the global node g5.

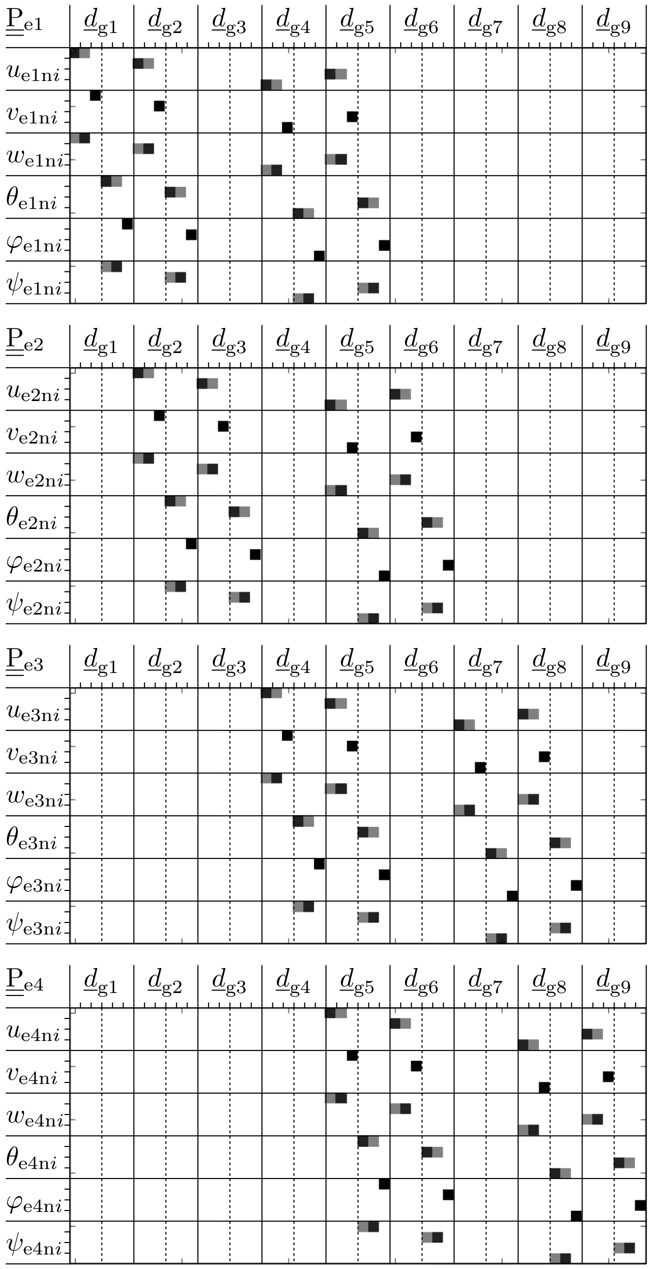

Figure 3: A grayscale representation of the terms of the four $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$ mapping matrices associated the elements of Fig. 1. The colormap spans from white (zero) to black (one); the lighter and the darker grey colors represent terms that equate in modulus $\sin 30°$ and $\cos 30°$, respectively.

the coefficients of the linear combination of global DOFs that equates the i local DOF of the j-th element; an example is proposed in the following to illustrate such relation.

With reference to the structure of Figure 1, w_{e1n2} and θ_{e1n1} respectively represent the 10th and the 13th local degrees of freedom of element 1.

Their global representation involves a subset of the g2 and g1 global nodes DOFs, respectively, namely

$$w_{\mathrm{e1n2}} = \langle \hat{k}_{\mathrm{e1}}, \hat{\imath}_{\mathrm{g2}} \rangle u_{\mathrm{g2}} + \langle \hat{k}_{\mathrm{e1}}, \hat{\jmath}_{\mathrm{g2}} \rangle v_{\mathrm{g2}} + \langle \hat{k}_{\mathrm{e1}}, \hat{k}_{\mathrm{g2}} \rangle w_{\mathrm{g2}} \tag{7}$$

$$\theta_{\mathrm{e1n1}} = \langle \hat{\imath}_{\mathrm{e1}}, \hat{\imath}_{\mathrm{g1}} \rangle \theta_{\mathrm{g1}} + \langle \hat{\imath}_{\mathrm{e1}}, \hat{\jmath}_{\mathrm{g1}} \rangle \phi_{\mathrm{g1}} + \langle \hat{\imath}_{\mathrm{e1}}, \hat{k}_{\mathrm{g1}} \rangle \psi_{\mathrm{g1}} \tag{8}$$

where $\hat{\imath}_{\mathrm{e1}}$,$\hat{\jmath}_{\mathrm{e1}}$, $\hat{k}_{\mathrm{e1}}$ are the orientation vectors of the element 1 local reference system, $\hat{\imath}_{\mathrm{g1}}$,$\hat{\jmath}_{\mathrm{g1}}$,$\hat{k}_{\mathrm{g1}}$ and $\hat{\imath}_{\mathrm{g2}}$,$\hat{\jmath}_{\mathrm{g2}}$,$\hat{k}_{\mathrm{g2}}$ are the orientation vectors of the global nodes 1 and 2 reference systems, and $\langle \cdot, \cdot \rangle$ represents their mutual scalar product, or, equivalently, the cosinus of the angle between two unit vectors.

The 10th and the 13th row of the $\underline{\underline{\mathrm{P}}}_{\mathrm{e1}}$ mapping matrix are defined based on Eqs.7 and 8, respectively, and they are null except for the elements

$$\begin{aligned}
\left[\underline{\underline{\mathrm{P}}}_{\mathrm{e1}}\right]_{10,7} &= \langle \hat{k}_{\mathrm{e1}}, \hat{\imath}_{\mathrm{g2}} \rangle & \qquad \left[\underline{\underline{\mathrm{P}}}_{\mathrm{e1}}\right]_{13,4} &= \langle \hat{\imath}_{\mathrm{e1}}, \hat{\imath}_{\mathrm{g1}} \rangle \\
\left[\underline{\underline{\mathrm{P}}}_{\mathrm{e1}}\right]_{10,8} &= \langle \hat{k}_{\mathrm{e1}}, \hat{\jmath}_{\mathrm{g2}} \rangle & \qquad \left[\underline{\underline{\mathrm{P}}}_{\mathrm{e1}}\right]_{13,5} &= \langle \hat{\imath}_{\mathrm{e1}}, \hat{\jmath}_{\mathrm{g1}} \rangle \\
\left[\underline{\underline{\mathrm{P}}}_{\mathrm{e1}}\right]_{10,9} &= \langle \hat{k}_{\mathrm{e1}}, \hat{k}_{\mathrm{g2}} \rangle & \qquad \left[\underline{\underline{\mathrm{P}}}_{\mathrm{e1}}\right]_{13,6} &= \langle \hat{\imath}_{\mathrm{e1}}, \hat{k}_{\mathrm{g1}} \rangle,
\end{aligned}$$

being u_{g2},v_{g2},w_{g2},θ_{g1},ϕ_{g1} and ψ_{g1} the 7th, 8th, 9th, 4th, 5th and 6th global degrees of freedom according to their position in $\underline{\mathrm{d}}_{\mathrm{g}}$.

Figure 3 presents a grayscale representation of the four $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$ matrices; please note the extremely sparse nature of those matrices, whose number of nonzero terms scales with the single element DOF cardinality, whereas the total number of terms scale with the whole structure DOF cardinality.

The rows of the rectangular $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$ mapping matrix are mutually orthonormal; the mapping matrix is orthogonal in the sense of the Moore-Penrose pseudoinverse, since its transpose and its pseudoinverse coincide.

The elemental mapping $\underline{\underline{\mathrm{P}}}_{\mathrm{e}j}$ matrices constitute an artifice that plays a double role in the local to global DOF mapping; if on one side the j-th element DOFs may be derived from their global counterpart as

$$\underline{\mathrm{d}}_{\,\mathrm{e}j} = \underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}\,\underline{\mathrm{d}}_{\,\mathrm{g}}, \tag{9}$$

on the other, the nodal actions expressed according to the local DOF system may be translated to the global scale by resorting to the transpose (which embodies the pseudoinverse) of the mapping matrix, as in

$$\underline{\mathrm{G}}_{\,\mathrm{g}\leftarrow\mathrm{e}j} = \underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}^{\top}\,\underline{\mathrm{F}}_{\,\mathrm{e}j}. \tag{10}$$

Based on 1, 9 and 10, the contribution of the j-th element to the elastic response of the structure may finally be described as the vector of global force components

$$\underline{\underline{\mathrm{F}}}_{\,\mathrm{g}\leftarrow\mathrm{e}j} = \underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}^{\top}\,\underline{\underline{\mathrm{K}}}_{\,\mathrm{e}j}\,\underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}\,\underline{\mathrm{d}}_{\,\mathrm{g}}; \tag{11}$$

that have to be applied at the structure DOFs in order to equilibrate the elastic reactions that arise at the nodes of the j-th element, if a deformed configuration is prescribed for the latter according to the $\underline{\mathrm{d}}_{\,\mathrm{g}}$ global displacement mode.

By accumulating the contribution of the various elements in a structure, the overall relation is obtained

$$\underline{\underline{\mathrm{F}}}_{\,\mathrm{g}} = \sum_j \underline{\underline{\mathrm{F}}}_{\,\mathrm{g}\leftarrow\mathrm{e}j} = \left(\sum_j \underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}^{\top}\,\underline{\underline{\mathrm{K}}}_{\,\mathrm{e}j}\,\underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}\right)\underline{\mathrm{d}}_{\,\mathrm{g}} = \underline{\underline{\mathrm{K}}}_{\,\mathrm{g}}\,\underline{\mathrm{d}}_{\,\mathrm{g}}, \tag{12}$$

that defines the $\underline{\underline{\mathrm{K}}}_{\,\mathrm{g}}$ global stiffness matrix as an assembly of the elemental contributions. The contribute accumulation at each summatory step is graphically represented in Fig. 4, in the case of the example structure of Fig. 1.

The global stiffness matrix is symmetric, and it shows nonzero terms at cells whose row and column indices are associate to two DOFs that are bridged by a direct elastic link – i.e., an element exists, that insists on both the nodes those DOFs pertain; since only a limited number of elements insist on each given node, the matrix is sparse, as shown in Fig. 4d.

An favourable numbering of the global nodes may be searched for, such that the nonzero terms are clustered within a (possibly) narrow band around the diagonal; the resulting stiffness matrix is hence *banded*, condition this that reduces both the storage memory requirements, and the computational effort in applying the various algebraic operators to the matrix.

The stiffness matrix (half-)bandwidth may be predicted by evaluating the bandwidth required for storing each element contribution

$$b_{\mathrm{e}j} = (i_{\max} - i_{\min} + 1)\, l, \tag{13}$$

and retaining the

$$b = \max_{\mathrm{e}j} b_{\mathrm{e}j} \tag{14}$$

peak value; in the formula 13, l is the number of DOF per element node, whereas $i_{\max}$ and $i_{\max}$ are the extremal integer labels associated to the element nodes, according to the global numbering.

0.1.3 External forces assembly

The element vector forces are accumulated to derive global external forces vector $\underline{\mathrm{F}}_{\,\mathrm{g}}$, as in

$$\underline{\mathrm{F}}_{\,\mathrm{g}} = \sum_j \underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}^{\top}\, \underline{\mathrm{F}}_{\,\mathrm{e}j}; \tag{15}$$

the transposed $\underline{\underline{\mathrm{P}}}_{\,\mathrm{e}j}^{\top}$ mapping matrix is employed to translate the actions on the local DOFs to their global counterpart.

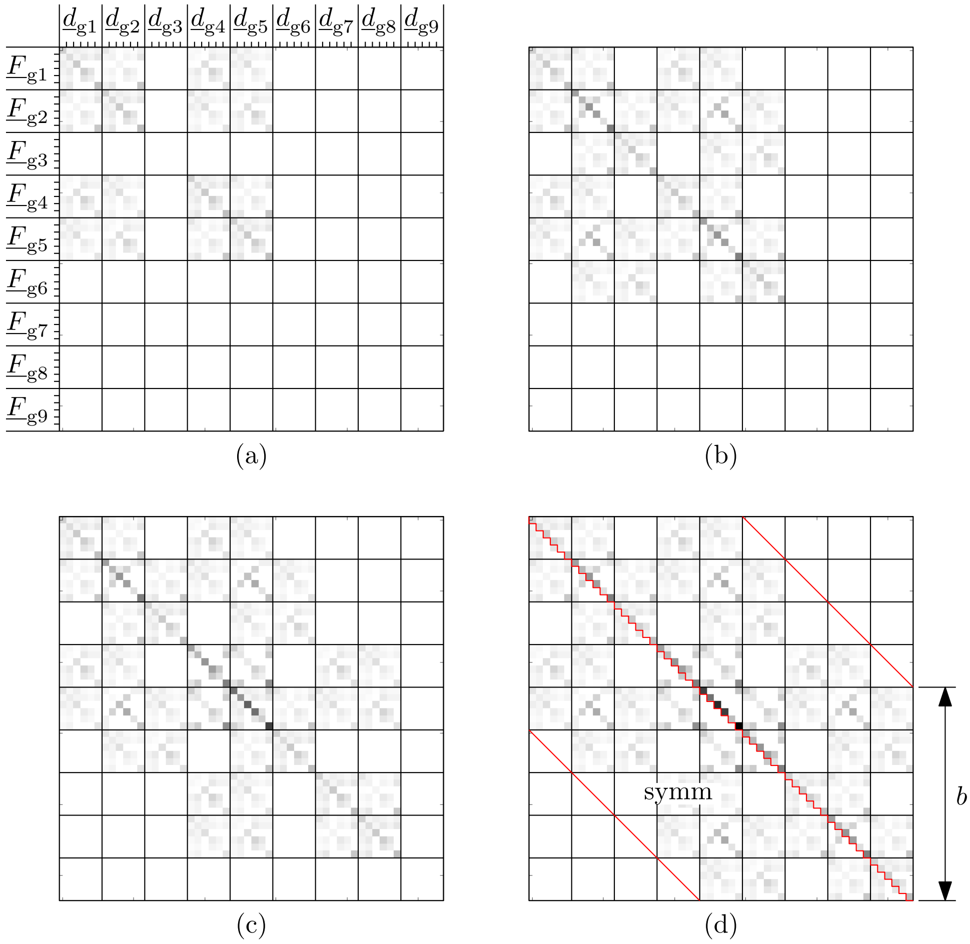

Figure 4: Graphical representation of the assembly steps for the stiffness matrix of the Fig. 1 structure. The zero-initialized form for the matrix that precedes the (a) step is omitted.