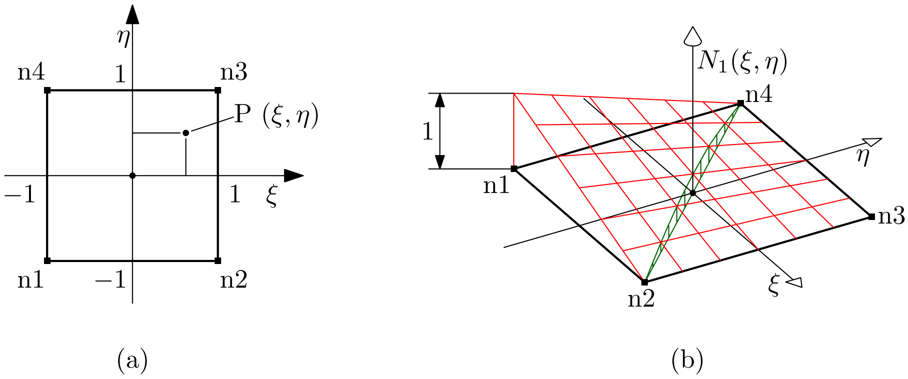

Figure 1: Quadrilateral elementary domain (a), and a representative weight function (b).

0.1 Preliminary results

0.1.1 Interpolation functions for the quadrilateral domain

The elementary quadrilateral domain. A quadrilateral domain is considered whose vertices are conventionally located at the $(\pm 1, \pm 1)$ points of an adimensional (ξ, η) plane coordinate system, see Figure 1. Scalar values f_i are associated to a set of *nodal* points $\mathrm{P}_i \equiv [\xi_i, \eta_i]$, which for the present case coincide with the quadrangle vertices, numbered as in Figure.

A $f(\xi, \eta)$ interpolation function may be devised by defining a set of nodal influence functions $N_i(\xi, \eta)$ to be employed as the coefficients (weights) of a moving weighted average

$$f(\xi, \eta) \stackrel{\text{def}}{=} \sum_i N_i(\xi, \eta) f_i \tag{1}$$

Requisites for such weight functions are:

- for each point of the domain, the sum of the weights is unitary

$$\sum_i N_i(\xi, \eta) = 1, \ \forall [\xi, \eta] \tag{2}$$

- to grant continuity of the $f(\xi, \eta)$ function with the nodal samples, the influence of a node is unitary at its location, whereas the influence of the others vanishes there, i.e.

$$N_i(\xi_j, \eta_j) = \delta_{ij} \tag{3}$$

where δ_{ij} is the Kronecker delta function.

Moreover, suitable functions should be continuous and straightforwardly differentiable up to any required degree.

Low order polynomials are ideal candidates for the application; for the particular domain, the nodal weight functions may be stated as

$$N_i(\xi, \eta) \stackrel{\text{def}}{=} \frac{1}{4}\,(1 \pm \xi)\,(1 \pm \eta)\,, \tag{4}$$

where sign ambiguity is resolved for each i-th node by enforcing Eqn. 3.

The bilinear interpolation function defined by Eqs. 1 and 4 turns into a general linear relation with (ξ, η) if the four sample points (ξ_i, η_i, f_i) are coplanar – but otherwise arbitrary – in the ξ, η, f space.

Further generality may be introduced by *not* enforcing coplanarity.

The weight functions for the four-node quadrilateral are in fact quadratic although incomplete[1] in nature, due to the presence of the $\xi\eta$ product, and the absence of any ξ^2, η^2 term.

Each $N_i(\xi, \eta)$ term, and the combined $f(\xi, \eta)$ function, defined as in Eqn. 1, behave linearly if restricted to ξ = const. or η = const. loci – and in particular along the four edges; quadratic behaviour may instead arise along a general direction, e.g. along the diagonals, as in Fig. 1b example. Such behaviour is called *bilinear*.

We now consider the $f(\xi, \eta)$ interpolation function partial derivatives. The partial derivative

$$\frac{\partial f}{\partial \xi} = \underbrace{\left(\frac{f_2 - f_1}{2}\right)}_{[\Delta f/\Delta \xi]_{12}} \underbrace{\left(\frac{1-\eta}{2}\right)}_{N_1+N_2} + \underbrace{\left(\frac{f_3 - f_4}{2}\right)}_{[\Delta f/\Delta \xi]_{43}} \underbrace{\left(\frac{1+\eta}{2}\right)}_{N_4+N_3} = a\eta + b \tag{5}$$

linearly varies in η from the incremental ratio value measured at the $\eta = -1$ lower edge, to the value measured at the $\eta = 1$ upper edge; the other partial derivative

$$\frac{\partial f}{\partial \eta} = \left(\frac{f_4 - f_1}{2}\right)\left(\frac{1-\xi}{2}\right) + \left(\frac{f_3 - f_2}{2}\right)\left(\frac{1+\xi}{2}\right) = c\xi + d. \tag{6}$$

[1] or, equivalently, *enriched linear*, as discussed above and in the following

behaves similarly, with $c = a$. Partial derivatives in ξ, η remain constant while moving along the corresponding differentiation direction[2].

The general quadrilateral domain. The interpolation functions introduced above for the natural quadrilateral may be profitably employed in defining a coordinate mapping between a general quadrangular domain – see Fig. 2a – and its reference counterpart, see Figures 1 or 2b.

In particular, we first define the $\underline{\xi}_i \mapsto \underline{\mathrm{x}}_i$ coordinate mapping for the four vertices[3] alone, where ξ, η are the reference (or natural, or elementary) coordinates and x, y are their physical counterpart.

Then, a mapping for the inner points may be derived by interpolation, namely

$$\underline{\mathrm{x}}\left(\underline{\xi}\right) = \underline{\mathrm{m}}\left(\underline{\xi}\right) = \sum_{i=1}^{4} N_i\left(\underline{\xi}\right)\underline{\mathrm{x}}_i, \tag{7}$$

or, by expliciting the $\underline{\mathrm{m}} \equiv \underline{\mathrm{x}}$ components,

$$\underline{\mathrm{m}}\left(\underline{\xi}\right) = \left[\begin{array}{c} x(\xi,\eta) \\ y(\xi,\eta) \end{array}\right]$$

with

$$x(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)x_i \qquad\qquad y(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)y_i.$$

The availability of an inverse $\underline{\mathrm{m}}^{-1} : \underline{\mathrm{x}} \mapsto \underline{\xi}$ mapping is not granted; in particular, a closed form representation for such inverse is not generally available[4].

In the absence of an handy inverse mapping function, it is convenient to reinstate the interpolation procedure obtained for the natural

[2] The relevance of such partial derivative orders will appear clearer to the reader once the strain field will have been derived in paragraph XXX.

[3] The condensed notation $\underline{\xi}_i \equiv (\xi_i, \eta_i)$, $\underline{\mathrm{x}}_i \equiv (x_i, y_i)$ for coordinate vectors is employed.

[4] Inverse relations are derived in [1], which however are case-defined and based on a selection table; for a given $\underline{\bar{\mathrm{x}}}$ physical point, however, Newton-Raphson iterations rapidly converge to the $\underline{\bar{\xi}} = \underline{\mathrm{m}}^{-1}\left(\underline{\bar{\mathrm{x}}}\right)$ solution if the centroid is chosen for algorithm initialization, see Section XXX

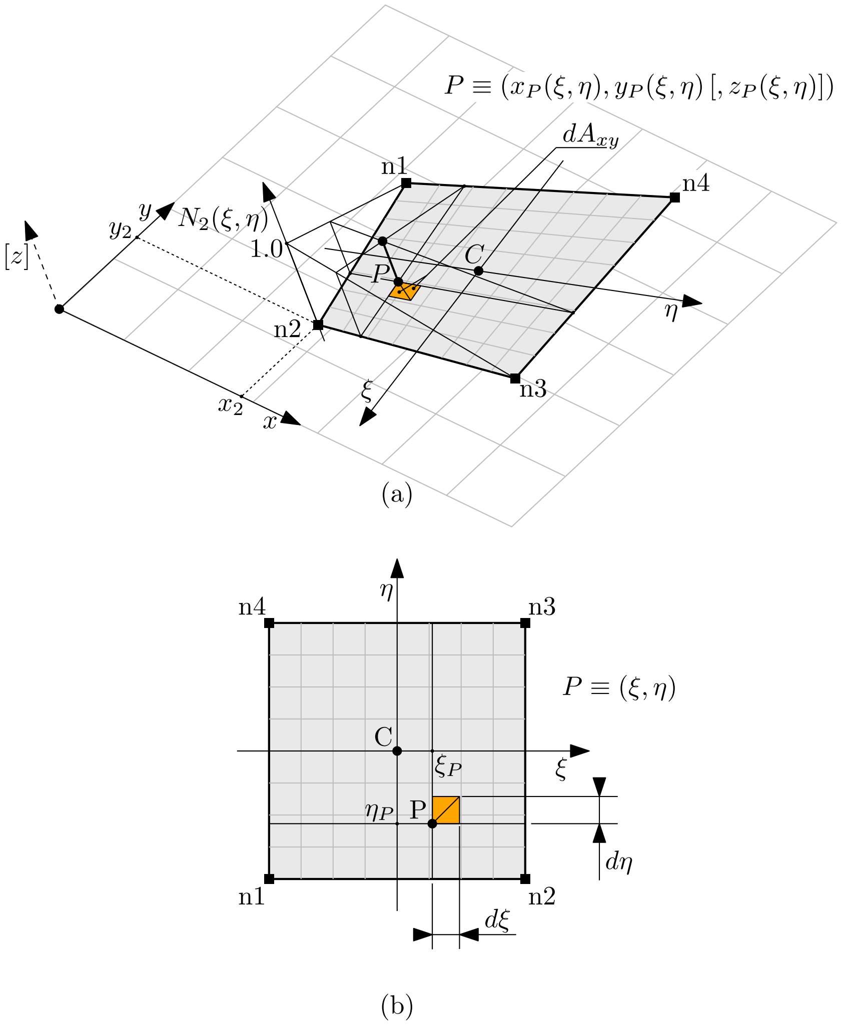

Figure 2: Quadrilateral general domain, (a), and its reference counterpart (b). If the general quadrangle is defined within a spatial environment, and not as a figure lying on the xy plane, limited z_i offsets are allowed at nodes with respect to such plane, which are not considered in Figure.

domain, i.e.

$$f(\xi,\eta) \stackrel{\text{def}}{=} \sum_i N_i(\xi,\eta) f_i \tag{8}$$

The four f_i nodal values are interpolated based on the *natural* ξ, η coordinates of an inner P point, and not as a function of its physical x, y coordinates, that are never promoted to the independent variable role.

The interpolation scheme behind the $\underline{\mathrm{m}}$ mapping – and the mapping itself – behaves linearly along η =const. and ξ =const. one dimensional subdomains, and in particular along the quadrangle edges[5]; the inverse mapping $\underline{\mathrm{m}}^{-1}$ exists and it is a linear function[6] along the image of those line segments on the physical plane, under the further condition that its length is nonzero[7]. Being a composition of linear functions, the interpolation function $f(\underline{\mathrm{m}}^{-1}(x,y))$ is also linear along the aforementioned subdomains, and in particular along the quadrangle edges.

The directional derivatives of f with respect to x or y are obtained based the indirect relation

$$\begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{\underline{\underline{\mathrm{J}}}'(\xi,\eta)} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \tag{9}$$

.

[5]see paragraph XXX

[6]A constructive proof may be defined for each edge as follows. We consider a generic Q point along such edge whose physical coordinates are (x_Q, y_Q). Of the two natural coordinates of Q, one is trivial to be derived since its value is constant along the edge. The other, for which we employ the λ placeholder symbol, may be defined through the expression

$$\lambda = 2\frac{(x_Q - x_i)(x_j - x_i) + (y_Q - y_i)(y_j - y_i)}{(x_j - x_i)^2 + (y_j - y_i)^2} - 1,$$

where i,j are the two subdomain endpoints at which λ equates -1 and $+1$, respectively, and $(x_i, y_i), (x_j, y_j)$ the associated physical coordinates. A similar function may be defined for any segment for which either ξ or η is constant, and not only for the quadrangle edges. Please note that the above inverse mapping formula is not applicable if and only if (IIF) the segment physical length at the denominator is zero.

[7]The case exists of an edge whose endpoints are superposed, i.e. the edge collapses to a point.

The function derivatives with respect to ξ, η are obtained as

$$\begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = \sum_i \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} f_i. \tag{10}$$

The *transposed* Jacobian matrix of the mapping function that appears in 9 is

$$\underline{\underline{J}}'(\xi, \eta) = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{11}$$

$$= \sum_i \left(\begin{bmatrix} \frac{\partial N_i}{\partial \xi} & 0 \\ \frac{\partial N_i}{\partial \eta} & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 & \frac{\partial N_i}{\partial \xi} \\ 0 & \frac{\partial N_i}{\partial \eta} \end{bmatrix} y_i \right) \tag{12}$$

If the latter matrix is assumed nonsingular – condition, this, that pairs the bijective nature of the $\underline{m}$ mapping, equation 9 may be inverted, thus leading to the form

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \left(\underline{\underline{J}}'\right)^{-1} \begin{bmatrix} \cdots & \frac{\partial N_i}{\partial \xi} & \cdots \\ \cdots & \frac{\partial N_i}{\partial \eta} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ f_i \\ \vdots \end{bmatrix}, \tag{13}$$

where the inner mechanics of the matrix-vector product are appointed for the Eq. 10 summation.

0.1.2 Gaussian quadrature rules for some relevant domains.

Reference one dimensional domain. The gaussian quadrature rule for approximating the definite integral of a $f(\xi)$ function over the $[-1, 1]$ reference interval is constructed as the customary weighted sum of internal function samples, namely

$$\int_{-1}^{1} f(\xi) d\xi \approx \sum_{i=1}^{n} f(\xi_i) w_i; \tag{14}$$

Its peculiarity is to employ location-weight pairs (ξ_i, w_i) that are optimal with respect to the polynomial class of functions. Nevertheless, such choice has revealed itself to be robust enough for for a more general employment.

Let's consider a m-th order polynomial

$$p(\xi) \stackrel{\text{def}}{=} a_m\xi^m + a_{m-1}\xi^{m-1} + \ldots + a_1\xi + a_0$$

whose exact integral is

$$\int_{-1}^{1} p(\xi)d\xi = \sum_{j=0}^{m} \frac{(-1)^j + 1}{j+1} a_j$$

The integration residual between the exact definite integral and the weighted sample sum is defined as

$$r\left(a_j, (\xi_i, w_i)\right) \stackrel{\text{def}}{=} \sum_{i=1}^{n} p(\xi_i)w_i - \int_{-1}^{1} p(\xi)d\xi \qquad (15)$$

The optimality condition is stated as follows: the quadrature rule involving n sample points (ξ_i, w_i), $i = 1\ldots n$ is optimal for the m-th order polynomial if a) the integration residual is null for general a_j values , and b) such condition does not hold for any lower-order sampling rule.

Once observed that the zero residual requirement is satisfied by any sampling rule if the polynomial a_j coefficients are all null, condition a) may be enforced by imposing that such zero residual value remains constant with varying a_j terms, i.e.

$$\left\{ \frac{\partial r\left(a_j, (\xi_i, w_i)\right)}{\partial a_j} = 0, \quad j = 0 \ldots m \right. \qquad (16)$$

A system of $m + 1$ polynomial equations of degree[8] $m + 1$ is hence obtained in the $2n$ (ξ_i, w_i) unknowns; in the assumed absence of redundant equations, solutions do not exist if the constraints outnumber the unknowns, i.e. $m > 2n - 1$. Limiting our discussion to the threshold condition $m = 2n-1$, an attentive algebraic manipulation of Eqns. 16 may be performed in order to extract the (ξ_i, w_i) solutions, which are unique apart from the pair permutations[9].

[8] the $(m + 1)$-th order $w_m\xi^m$ term appears in equations

[9] In this note, location-weight pairs are obtained for the gaussian quadrature rule of order $n = 2$, aiming at illustrating the general procedure. The general

n	ξ_i	w_i
1	0	2
2	$\pm\frac{1}{\sqrt{3}}$	1
3	0	$\frac{8}{9}$
	$\pm\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
4	$\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18+\sqrt{30}}{36}$
	$\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18-\sqrt{30}}{36}$

Table 1: Integration points for the lower order gaussian quadrature rules.

Eqns. 16 solutions are reported in Table 1 for quadrature rules with up to $n = 4$ sample points[10].

It is noted that the integration points are symmetrically distributed

$m = 2n - 1 = 3$rd order polynomial is stated in the form

$$p(\xi) = a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0, \qquad \int_{-1}^{1} p(\xi)d\xi = \frac{2}{3}a_2 + 2a_0,$$

whereas the integral residual is

$$r = a_3\left(w_1\xi_1^3 + w_2\xi_2^3\right) + a_2\left(w_1\xi_1^2 + w_2\xi_2^2 - \frac{2}{3}\right) + a_1\left(w_1\xi_1 + w_2\xi_2\right) + a_0\left(w_1 + w_2 - 2\right)$$

Eqns 16 may be derived as

$$\begin{cases} 0 = \frac{\partial r}{\partial a_3} = w_1\xi_1^3 + w_2\xi_2^3 & (e_1) \\ 0 = \frac{\partial r}{\partial a_2} = w_1\xi_1^2 + w_2\xi_2^2 - \frac{2}{3} & (e_2) \\ 0 = \frac{\partial r}{\partial a_1} = w_1\xi_1 + w_2\xi_2 & (e_3) \\ 0 = \frac{\partial r}{\partial a_0} = w_1 + w_2 - 2 & (e_4) \end{cases}$$

which are independent of the a_j coefficients.

By composing $\left(e_1 - \xi_1^2 e_3\right)/(w_2\xi_2)$ it is obtained that $\xi_2^2 = \xi_1^2$; e_2 may then be written as $(w_1 + w_2)\xi_1^2 = 2/3$, and then as $2\xi_1^2 = 2/3$, according to e_4. It derives that $\xi_{1,2} = \pm 1/\sqrt{3}$. Due to the opposite nature of the roots, e_3 implies $w_2 = w_1$, relation, this, that turns e_4 into $2w_1 = 2w_2 = 2$, and hence $w_{1,2} = 1$.

[10]see `https://pomax.github.io/bezierinfo/legendre-gauss.html` for higher order gaussian quadrature rule sample points.

with respect to the origin, and that the function is never sampled at the $\{-1, 1\}$ extremal points.

General one dimensional domain. The extension of the one dimensional quadrature rule from the reference domain $[-1, 1]$ to a general $[a, b]$ domain is pretty straightforward, requiring just a change of integration variable – i.e. a mapping function $x = m(\xi)$ s.t. $a = m(-1)$ and $b = m(1)$ – to obtain the following

$$\int_a^b g(x)dx = \int_{-1}^{1} g\left(m(\xi)\right) \frac{dm}{d\xi} d\xi \approx \sum_{i=1}^{n} g\left(m(\xi_i)\right) \left.\frac{dm}{d\xi}\right|_{\xi=\xi_i} w_i. \quad (17)$$

Such a mapping function may be conveniently defined along the same lines as the weight (or shape) function based interpolation, thus obtaining

$$m(x) = \underbrace{\left(\frac{1-\xi}{2}\right)}_{N_1} a + \underbrace{\left(\frac{1+\xi}{2}\right)}_{N_2} b.$$

The first order derivative may be evaluated as

$$\frac{dm}{d\xi} = \frac{dN_1}{d\xi} a + \frac{dN_2}{d\xi} b = \frac{b-a}{2}$$

and it is constant along the interval, so that it may be collected outside of the summation, thus leading to

$$\int_a^b g(x)dx \approx \frac{b-a}{2} \sum_{i=1}^{n} g\left(\frac{b+a}{2} + \frac{b-a}{2}\xi_i\right) w_i. \quad (18)$$

Reference quadrangular domain. A quadrature rule for the reference quadrangular domain of Figure 1a may be derived by nesting the quadrature rule defined for the reference interval, see Eqn. 14, thus obtaining

$$\int_{-1}^{1}\int_{-1}^{1} f\left(\xi, \eta\right) d\xi d\eta \approx \sum_{i=1}^{p}\sum_{j=1}^{q} f\left(\xi_i, \eta_j\right) w_i w_j \quad (19)$$

where (ξ_i, w_i) and (η_j, w_j) are the coordinate-weight pairs of the two quadrature rules of p and q order, respectively, employed for spanning the two coordinate axes. The equivalent notation

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta \approx \sum_{l=1}^{pq} f\left(\underline{\xi}_{\,l}\right) w_l \tag{20}$$

emphasises the characteristic nature of the pq point/weight pairs for the domain and for the quadrature rule employed; a general integer bijection[11] $\{1\ldots pq\} \leftrightarrow \{1\ldots p\} \times \{1\ldots q\}$, $l \leftrightarrow (i,j)$ may be utilized to formally derive the two-dimensional quadrature rule pairs

$$\underline{\xi}_{\,l} = (\xi_i, \eta_j), \quad w_l = w_i w_j, \quad l = 1\ldots pq \tag{21}$$

from their uniaxial counterparts.

General quadrangular domain. The rectangular infinitesimal area $dA_{\xi\eta} = d\xi d\eta$ in the neighborhood of a ξ_P, η_P location, see Figure 2b, is mapped to the dA_{xy} quadrangle of Figure 2a, which is composed by the two triangular areas

$$\begin{aligned} dA_{xy} = {} & \frac{1}{2!}\begin{vmatrix} 1 & x(\xi_P, \eta_P) & y(\xi_P, \eta_P) \\ 1 & x(\xi_P + d\xi, \eta_P) & y(\xi_P + d\xi, \eta_P) \\ 1 & x(\xi_P + d\xi, \eta_P + d\eta) & y(\xi_P + d\xi, \eta_P + d\eta) \end{vmatrix} + \\ & + \frac{1}{2!}\begin{vmatrix} 1 & x(\xi_P + d\xi, \eta_P + d\eta) & y(\xi_P + d\xi, \eta_P + d\eta) \\ 1 & x(\xi_P, \eta_P + d\eta) & y(\xi_P, \eta_P + d\eta) \\ 1 & x(\xi_P, \eta_P) & y(\xi_P, \eta_P) \end{vmatrix}. \end{aligned} \tag{22}$$

[11] e.g.

$$\{i-1; j-1\} = (l-1) \bmod (p,q), \quad l-1 = (j-1)q + (i-1)$$

where the operator

$$\{a_n; \ldots; a_3; a_2; a_1\} = m \bmod (b_n, \ldots, b_3, b_2, b_1)$$

consists in the extraction of the n least significant a_i digits of a mixed radix representation of the integer m with respect to the sequence of b_i bases, with $i = 1\ldots n$.

The determinant formula for the area of a triangle, shown below along with its n-dimensional symplex hypervolume generalization,

$$\mathcal{A} = \frac{1}{2!}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, \quad \mathcal{H} = \frac{1}{n!}\begin{vmatrix} 1 & \underline{\mathrm{x}}_1 & \\ 1 & \underline{\mathrm{x}}_2 & \\ \vdots & \vdots & \\ 1 & \underline{\mathrm{x}}_{n+1} & \end{vmatrix} \tag{23}$$

has been employed above.

By operating a local multivariate linearization of the 22 matrix terms, the relation

$$\begin{aligned} dA_{xy} \approx \quad & \frac{1}{2!}\begin{vmatrix} 1 & x & y \\ 1 & x + x_{,\xi}d\xi & y + y_{,\xi}d\xi \\ 1 & x + x_{,\xi}d\xi + x_{,\eta}d\eta & y + y_{,\xi}d\xi + y_{,\eta}d\eta \end{vmatrix} + \\ & + \frac{1}{2!}\begin{vmatrix} 1 & x + x_{,\xi}d\xi + x_{,\eta}d\eta & y + y_{,\xi}d\xi + y_{,\eta}d\eta \\ 1 & x + x_{,\eta}d\eta & y + y_{,\eta}d\eta \\ 1 & x & y \end{vmatrix} \end{aligned}$$

is obtained, where $x, y, x_{,\xi}, x_{,\eta}, y_{,\xi}$, and $y_{,\eta}$ are the x, y functions and their first order partial derivatives, evaluated at the (ξ_P, η_P) point; infinitesimal terms of order higher than $d\xi, d\eta$ are neglected.

After some matrix manipulations[12], the following expression is ob-

[12] In the first determinant, the second row is subtracted from the third one, and the first row is subtracted from the second one. The $d\xi$, $d\eta$ factors are then collected from the second and the third row respectively. In the second determinant, the second row is subtracted from the first one, and the third row is subtracted from the second one. The $d\xi$, $d\eta$ factors are then collected from the first and the second row respectively. We now have

$$dA_{xy} = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 0 & x_{,\xi} & y_{,\xi} \\ 0 & x_{,\eta} & y_{,\eta} \end{vmatrix} d\xi d\eta + \frac{1}{2}\begin{vmatrix} 0 & x_{,\xi} & y_{,\xi} \\ 0 & x_{,\eta} & y_{,\eta} \\ 1 & x & y \end{vmatrix} d\xi d\eta$$

The first column of both the determinants contains a single, unitary, nonzero term, whose row and column indexes are even once added up; the determinants of the associated complementary minors hence equate their whole matrix counterpart. We hence obtain

$$dA_{xy} = \frac{1}{2}\begin{vmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{vmatrix} d\xi d\eta + \frac{1}{2}\begin{vmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{vmatrix} d\xi d\eta$$

from which Eq.24 may be straightforwardly derived.

tained

$$dA_{xy} = \underbrace{\begin{vmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{vmatrix}}_{|J^{\mathrm{T}}(\xi_P,\eta_P)|} dA_{\xi\eta} \tag{24}$$

that equates the ratio of the mapped and of the reference areas to the determinant of the transformation (transpose) Jacobian matrix[13].

After the preparatory passages above, we obtain

$$\iint_{A_{xy}} g(x,y) dA_{xy} = \int_{-1}^{1}\int_{-1}^{1} g\left(x\left(\xi,\eta\right), y\left(\xi,\eta\right)\right) \left|J(\xi,\eta)\right| d\xi d\eta, \tag{25}$$

thus reducing the quadrature over a general domain to its reference domain counterpart, which has been discussed in the paragraph above.

Based on Eqn. 20, the quadrature rule

$$\iint_{A_{xy}} g(\underline{x}) dA_{xy} \approx \sum_{l=1}^{pq} g\left(\underline{x}\left(\underline{\xi}_l\right)\right) \left|J(\underline{\xi}_l)\right| w_l \tag{26}$$

is derived, stating that the definite integral of a g integrand over a quadrangular domain pertaining to the physical x, y plane (x, y are dimensional quantities, namely lengths) may be approximated as follows:

1. a reference-to-physical domain mapping is defined, that is based on the vertex physical coordinate interpolation;
2. the function is sampled at the physical locations that are the images of the Gaussian integration points previously obtained for the reference domain;
3. a weighted sum of the collected samples is performed, where the weights consist in the product of i) the adimensional w_l Gauss point weight (suitable for integrating on the reference domain), and ii) a dimensional area scaling term, that equals the determinant of the transformation Jacobian matrix, locally evaluated at the Gauss points.

[13]The Jacobian matrix for a general $\underline{\xi} \mapsto \underline{x}$ mapping is in fact defined according to

$$[J(\underline{\xi}_P)]_{ij} \overset{\text{def}}{=} \left.\frac{\partial x_i}{\partial \xi_j}\right|_{\underline{\xi}=\underline{\xi}_P} \qquad i,j = 1\ldots n$$

being i the generic matrix term row index, and j the column index

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